

# PLANE PARTITIONS WITH A “PIT”: GENERATING FUNCTIONS AND REPRESENTATION THEORY

M. BERSHTEIN, B. FEIGIN, G. MERZON

**ABSTRACT.** We study plane partitions satisfying condition  $a_{n+1,m+1} = 0$  (this condition is called “pit”) and asymptotic conditions along three coordinate axes. We find the formulas for generating function of such plane partitions.

Such plane partitions label the basis vectors in certain representations of quantum toroidal  $\mathfrak{gl}_1$  algebra, therefore our formulas can be interpreted as the characters of these representations. The resulting formulas resemble formulas for characters of tensor representations of Lie superalgebra  $\mathfrak{gl}_{m|n}$ . We discuss representation theoretic interpretation of our formulas using  $q$ -deformed  $W$ -algebra  $\mathfrak{gl}_{m|n}$ .

## 1. INTRODUCTION

In this paper we study certain problems of enumerative combinatorics of 3d Young diagrams, which are motivated by representation theory.

It is convenient to identify 3d Young diagrams with plane partitions i.e. collection of nonnegative integers  $a_{i,j}$  such that  $a_{i,j} \geq a_{i+1,j}$ ,  $a_{i,j} \geq a_{i,j+1}$  and all but finite number of  $a_{i,j}$  equals 0. Later we will also consider more general plane partitions.

Denote by  $|a| = \sum a_{i,j}$ , i.e. the number of boxes in the corresponding 3d Young diagram. For any set  $\mathcal{A}$  of plane partitions define its generating function by  $\sum_{a \in \mathcal{A}} q^{|a|}$ . Such functions were extensively studied in enumerative combinatorics, for example one of MacMahon’s formulas has the form

$$\sum_{\{a | a_{n+1,1}=0\}} q^{|a|} = q^{-\binom{n-1}{2}} \frac{V(1, q, \dots, q^{n-1})}{(q)_\infty^n},$$

where  $V(x_1, \dots, x_n) = \prod_{i < j} (x_i - x_j)$  is the Vandermonde product and by  $(q)_\infty$  we denote the product  $\prod_{k=1}^\infty (1 - q^k)$ . The limit  $n \rightarrow \infty$  gives well known MacMahon formula for the generating function of all plane partitions  $\sum_a q^{|a|} = \prod_{k=1}^\infty (1 - q^k)^{-k}$ .

We study plane partitions satisfying the condition

$$a_{n+1,m+1} = 0. \tag{1.1}$$

We will call such condition “pit” in box  $(n+1, m+1)$ . Moreover we will consider plane partitions  $a_{i,j}$  with infinite number of non-zero  $a_{i,j}$  and some of  $a_{i,j}$  equal to  $\infty$ , satisfying following asymptotic conditions

$$\mathbf{1.} \lim_{j \rightarrow \infty} a_{i,j} = \nu_i, \quad \mathbf{2.} \lim_{i \rightarrow \infty} a_{i,j} = \mu_j, \quad \mathbf{3.} a_{i,j} = \infty, \text{ iff } (i, j) \in \lambda, \tag{1.2}$$

where  $\nu, \mu, \lambda$  are partitions (see Fig. 1).

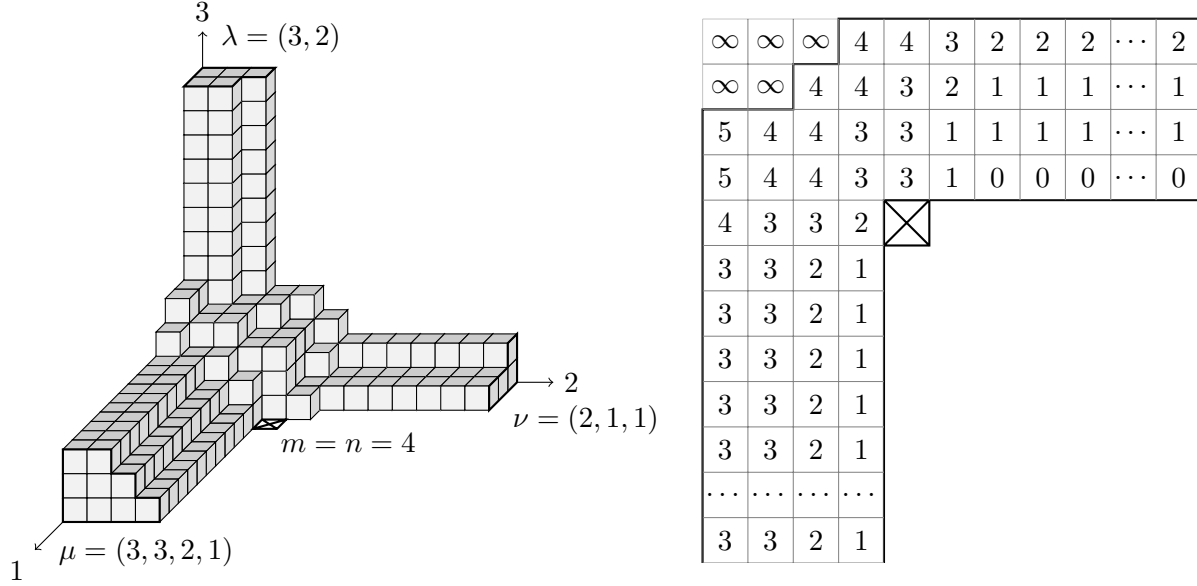


FIGURE 1.

We denote by  $\chi_{\lambda, \nu, \mu}^{n, m}(q)$  the generating function of plane partitions which satisfy (1.1), (1.2) (for the definition of  $|a|$  see (2.1)). It follows from these conditions that  $l(\nu) \leq n$ ,  $l(\mu) \leq m$ , and  $\lambda_{n+1} < m + 1$ .

Note that asymptotic conditions (1.2) appear in the theory of topological vertex [25]. We do not know string theory interpretation of (1.1), our motivation comes from representation theory, which we discuss below.

In some particular cases the formulas for functions  $\chi_{\lambda, \nu, \mu}^{n, m}(q)$  were known before. In order to write down the answers we need some notation. By  $\rho_n$  we denote the partition  $(n-1, n-2, \dots, 1, 0)$ . For any partition of no greater than  $n$  parts  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  by  $q^{\lambda + \rho_n}$  we denote  $(q^{\lambda_1 + n - 1}, \dots, q^{\lambda_n})$ . By  $a_{\lambda + \rho_n}(x_1, \dots, x_n)$  we denote the antisymmetric polynomial:

$$a_{\lambda + \rho_n}(x_1, \dots, x_n) = \det \left( x_i^{\lambda_j + n - j} \right). \quad (1.3)$$

The formula for  $\chi_{\lambda, \nu, \mu}^{n, m}(q)$  is known in the case  $m = 0$ , i.e. when the “pit” is located near the “wall”. Namely

$$\chi_{\emptyset, \nu, \lambda}^{n, 0}(q) = \frac{q^{\sum_{i=1}^n (\lambda_i + n - i)(\nu_i + n - i)}}{(q)_\infty^n} a_{\nu + \rho_n}(q^{-\lambda - \rho_n}), \quad (1.4)$$

see e.g. [7, Theorem 4.6]. Clearly this is a generalization of MacMahon formula given above. Another known case is where two asymptotic conditions vanishes  $\lambda = \mu = \emptyset$ , see [14], [15].

In our paper we find the formula for  $\chi_{\lambda, \nu, \mu}^{n, m}(q)$  in general case. Actually we prove three formulas, which are algebraically equivalent but have different form and meaning. They are given in Theorems 1, 2, 3 below. Here we give the simplest (but already new) particular

case of Theorem 2

$$\chi_{\mu,\nu,\emptyset}^{n,n}(q) = \sum_{A_1 > A_2 > \dots > A_n \geq 0} (-1)^{\sum_{i=1}^n A_i} q^{\sum_{i=1}^n \binom{A_i+1}{2}} \frac{a_{\nu+\rho_n}(q^A) a_{\mu+\rho_n}(q^{-A})}{(q)_\infty^{2n}}, \quad (1.5)$$

Note that each summand is a product of two expressions on the right side of (1.4).

Since our three formulas are algebraically equivalent it is enough to prove any of them. We give two different combinatorial proofs, one for Theorem 1 and one for Theorem 3. These proofs are simpler than ones of particular cases given in [14], [15].

The first proof is based on a bijection between plane partitions and collections of non crossing paths. The number of such collections is computed using Lindström–Gessel–Viennot lemma [20],[17]. Such proof gives a determinantal expression for  $\chi_{\lambda,\nu,\mu}^{n,m}(q)$ , see Theorem 1.

In the second proof we interpret conditions (1.1),(1.2) as a definition of certain infinite dimensional polyhedron. We compute the generating function of integer points in this polyhedron as a sum of contribution of vertices, using Brion theorem [4]. Such proof gives a “bosonic formula”<sup>1</sup> for  $\chi_{\lambda,\nu,\mu}^{n,m}(q)$ , see Theorem 3.

The conditions (1.1),(1.2) appeared in [9] in the context of representation theory of quantum toroidal algebra  $U_{\vec{q}}(\mathfrak{gl}_1)$ . Namely, plane partitions which satisfy these conditions label a basis in MacMahon modules  $\mathcal{N}_{\lambda,\nu,\mu}^{n,m}(v)$  over  $U_{\vec{q}}(\mathfrak{gl}_1)$ . Therefore  $\chi_{\lambda,\nu,\mu}^{n,m}(q)$  is the character of the representation  $\mathcal{N}_{\lambda,\nu,\mu}^{n,m}(v)$ . It is natural to ask for representation theoretic interpretation of our character formulas.

We claim that there exist resolutions of  $\mathcal{N}_{\lambda,\nu,\mu}^{n,m}(v)$  such that their Euler characteristics coincide with our character formulas. In such cases we say that resolution is a *materialization* of character formula. For example BGG resolution [3] is a materialization of Weyl character formula. Zelevinsky constructed complex which is a materialization of Jacobi–Trudi formula for Schur polynomials [29].

Our formulas for functions  $\chi_{\lambda,\nu,\mu}^{n,m}(q)$  resemble the formulas for characters of tensor representations of Lie superalgebra  $\mathfrak{gl}_{m|n}$ . This similarity can be explained by the fact that the representations  $\mathcal{N}_{\lambda,\nu,\mu}^{n,m}(v)$  are actually representations of certain  $q$ -deformed  $W$ -algebra, which we call  $W_{\vec{q}}(\mathfrak{gl}_{n|m})$ .

Such  $W$ -algebras appear as follows. There is an easy (but not written in the literature) fact that  $\mathcal{N}_{\lambda,\nu,\mu}^{n,m}(v)$  is a subfactor of Fock representation of  $U_{\vec{q}}(\mathfrak{gl}_1)$ . On such Fock modules image of  $U_{\vec{q}}(\mathfrak{gl}_1)$  commutes with certain operators, which are called screening operators. Algebras of elements which commute with screening operators are usually called  $W$ -algebras. The structure of screening operators in our case suggests the name  $W_{\vec{q}}(\mathfrak{gl}_{n|m})$ .

There exists conformal limit  $q \rightarrow 1$  of screening operators and we denote the algebra which commutes with them by  $W(\mathfrak{gl}_{n|m})$ . For  $m = 0$  this algebra coincides with the algebra  $W(\mathfrak{gl}_n)$  [6]. The algebras  $W(\mathfrak{gl}_{n|1})$  coincide with the  $W_n^{(2)}$  algebras introduced in [16]. We didn’t find reference for generic  $n, m$  (note that our  $W$ -algebras differ from ones introduced in [18]).

---

<sup>1</sup>Usually a formula is called bosonic if it equals a linear combination of characters of algebra of polynomials. In our case bosonic formula is a combination of terms  $q^\Delta / (q)_\infty^{n+m}$ .

Standard statement in the theory of vertex algebras is an equivalence of the abelian categories of certain representations of vertex algebra and certain representations of quantum group. This is a statement similar to Drinfeld–Kohno or Kazhdan–Lusztig theorem. We conjecture that under this equivalence  $W(\mathfrak{gl}_{n|m})$  is related to the product of quantum groups  $U_q \mathfrak{gl}_{n|m} \otimes U_{q'} \mathfrak{gl}_n \otimes U_{q''} \mathfrak{gl}_m$  for certain  $q, q', q''$ . And representations  $\mathcal{N}_{\lambda, \nu, \mu}^{n, m}(v)$  under this equivalence goes to the tensor products  $L_{\lambda}^{(n|m)} \otimes L_{\nu}^{(n)} \otimes L_{\mu}^{(m)}$ , where  $L_{\nu}^{(n)}$  and  $L_{\mu}^{(m)}$  are finite dimensional irreducible representations of  $U_{q'} \mathfrak{gl}_n$  and  $U_{q''} \mathfrak{gl}_m$  correspondingly and  $L_{\lambda}^{(n|m)}$  is a tensor representation of  $U_q \mathfrak{gl}_{n|m}$ .

**Plan of the paper.** In Section 2 we give precise statements of our main results for  $\chi_{\lambda, \nu, \mu}^{n, m}(q)$  with necessary notation and comments. The remaining sections 3, 4, 5 are independent of each other. Sections 3 and 4 are devoted to the combinatorial proofs based on Lindström–Gessel–Viennot lemma and Brion theorem correspondingly. Section 5 is devoted to the algebraic discussion, first we give a definition of appropriate  $W$ -algebras in terms of screening operators. Conjectural materializations of our character formulas are discussed in subsection 5.5, relation to quantum group  $U_q \mathfrak{gl}_{n|m} \otimes U_{q'} \mathfrak{gl}_n \otimes U_{q''} \mathfrak{gl}_m$  in subsection 5.7.

**Acknowledgments.** We thank A. Babichenko, E. Gorsky, A. Kirillov, A. Litvinov, I. Makhlin, G. Mutafyan, G. Olshanski, Y. Pugai, A. Sergeev for interest to our work and discussions.

The article was prepared within the framework of a subsidy granted to the National Research University Higher School of Economics by the Government of the Russian Federation for the implementation of the Global Competitiveness Program. M. B. acknowledges the financial support of Simons-IUM fellowship, RFBR grant mol\_a\_ved 15-32-20974 and Young Russian Math Contest.

## 2. MAIN RESULTS

**2.1.** Due to asymptotic conditions (1.2)  $a_{ij} \geq \nu_i$  and  $a_{ij} \geq \mu_j$ . In order to define the grading  $|a| \sim \sum a_{ij}$  we need to subtract these asymptotic values  $\nu_i, \mu_j$ .

We will use the following definition

$$|a| = \sum_{i-n \leq j-m, (i,j) \notin \lambda} (a_{ij} - \nu_i) + \sum_{i-n > j-m, (i,j) \notin \lambda} (a_{ij} - \mu_j) \quad (2.1)$$

Note that this definition of grading is not invariant under  $m, \mu \leftrightarrow n, \nu$  symmetry. Geometrically the definition (2.1) can be restated as follows. We draw a staircase line from the point  $(m, n)$  as on the picture below. This line divides the base of the plane partition  $a$  into two parts. We subtract  $\nu_i$  from cells in the upper part and  $\mu_j$  from cells in the left part, see Fig. 2.

Let  $r = \min\{t | \lambda_{n-t} \geq m - t\}$ ,  $0 \leq r \leq \min\{n, m\}$ . Geometrically  $r$  is the number of boxes above the staircase line starting from the point  $(m, n)$ . In the picture above we have  $r = 2$ . Note that this number  $r$  has a interpretation in terms of representation theory of  $\mathfrak{gl}(m|n)$  namely  $r$  is called the degree of atypicality of the tensor representation of  $\mathfrak{gl}(m|n)$  corresponding to  $\lambda$ .

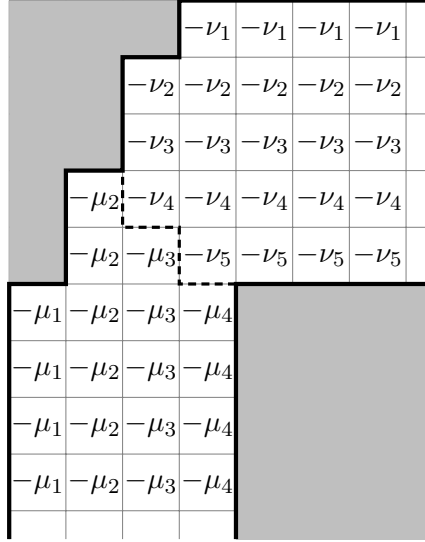


FIGURE 2.

In order to write down the formula for the generating function we parametrize  $\lambda$  by some analogue of Frobenius coordinates. We introduce two partitions  $\pi, \kappa$  by  $\pi_i = \lambda_i - (m - r)$  for  $i = 1, \dots, n - r$  and  $\kappa_j = \lambda'_j - (n - r)$  for  $j = 1, \dots, m - r$ , where  $\lambda'$  denotes transpose of the partition  $\lambda$ . We denote components of partitions  $\nu, \mu, \pi, \kappa$  shifted by  $\rho$  by the corresponding capital Latin letters:  $N_i = \nu_i + n - i$ ,  $M_j = \mu_j + m - j$ ,  $P_i = \pi_i + (n - r) - i$ ,  $Q_j = \kappa_j + (m - r) - j$ .

**2.2.** In the simplest case  $n = 1, m = 0$  for any asymptotic conditions  $\lambda, \nu$  the generating function of partitions  $\chi_{\nu, \emptyset, \lambda}^{1,0}(q)$  equals  $1/(q)_\infty$  (and similarly for  $n = 0, m = 1$  case).

Now consider the  $n = m = 1$  case. If  $\lambda \neq \emptyset$  then the plane partitions decompose into two partitions so the generating function equals  $1/(q)_\infty^2$ .

There is an clear bijection between our plane partitions and  $V$ -partitions [27] i.w. the  $\mathbb{N}$ -arrays of integer numbers:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_2 & \dots \\ a_0 & b_1 & b_2 & b_2 & \dots \end{pmatrix},$$

such that  $a_0 \geq a_1 \geq a_2 \geq \dots$ ,  $a_0 \geq b_1 \geq b_2 \geq \dots$ ,  $\lim_{i \rightarrow \infty} a_i = \nu_1$ ,  $\lim_{i \rightarrow \infty} b_i = \mu_1$ . The weight of  $V$ -partition is defined as

$$N = \sum_{i \geq 0} (a_i - a) + \sum_{i \geq 1} (b_i - b).$$

**Lemma 2.1.** *The generating function of  $V$ -partitions with asymptotic conditions  $\lim_{i \rightarrow \infty} a_i = \nu_1$ ,  $\lim_{i \rightarrow \infty} b_i = \mu_1$  equals*

$$R(d; q) := \sum_{i=0}^{\infty} (-1)^i \frac{q^{\frac{i(i+1)}{2}} q^{di}}{(q)_\infty^2}, \quad (2.2)$$

where  $d = \nu_1 - \mu_1$ .

This lemma can be proved by a kind of inclusion-exclusion argument. For the case  $\nu_1 = \mu_1$  see [27, Sec. 2.5]. The general case can be proved in a similar manner. See also [9, Cor. 5.6].

Now we can write down the first formula for  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$ .

**Theorem 1.** *The generating function  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  is equal to the determinant of a block matrix of the size  $(m+n-r) \times (m+n-r)$*

$$\chi_{\mu,\nu,\lambda}^{n,m}(q) = \frac{(-1)^{mn-r} q^{\Delta_{\mu,\nu,\lambda}^{m,n}}}{(q)_\infty^{m+n}} \det \begin{pmatrix} \left( \sum_{a \geq 0} (-1)^a q^{\binom{a+1}{2}} q^{(N_j - M_i)a} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} & (q^{-M_i Q_j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-r}} \\ (q^{-N_j(P_i+1)})_{\substack{1 \leq i \leq n-r \\ 1 \leq j \leq n}} & 0 \end{pmatrix}, \quad (2.3)$$

where  $\Delta_{\mu,\nu,\lambda}^{m,n} = \sum_{j=1}^{m-r} M_j Q_j + \sum_{i=1}^{n-r} N_i(P_i + 1)$ .

Clearly this formula generalizes previous consideration in the  $n, m \leq 1$  case, where determinant becomes  $1 \times 1$ . One can think that the formula (2.3) is similar to Jacobi–Trudi formula, which expresses generic Schur polynomial  $s_\lambda$  in terms of Schur polynomials corresponding to rows (or columns).

The Theorem 1 is proven in Section 3. It is natural that the determinant expression for the generating function can be proven using non-intersecting paths and Lindström–Gessel–Viennot lemma.

Let us mention two more special cases where we have only one block in the matrix. In the case of  $m = 0$  we have  $r = 0$ ,  $\pi = \lambda$  and after a multiplication on  $(q)_\infty^n$  the determinant becomes equal to  $a_{\nu+\rho_n}(q^{-\lambda-\rho_n})$ . So we get the known formula (1.4).

In the case  $m = n$  and  $\lambda = \emptyset$  the formula (2.3) simplifies to  $\det(R(N_j - M_i; q))$ . This formula was proven in [15] (following [9]) under the additional assumption that  $\mu = \emptyset$ .

**2.3.** The determinant in formula (2.3) can be calculated.

**Theorem 2.** *The generating function  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  is equal to the sum over  $r$ -tuples of integer numbers  $A_1 > A_2 > \dots > A_r \geq 0$*

$$\chi_{\mu,\nu,\lambda}^{n,m}(q) = (-1)^{r(m+n)} q^{\Delta_{\mu,\nu,\lambda}^{m,n}} \sum_{A_1 > A_2 > \dots > A_r \geq 0} (-1)^{\sum_{i=1}^r A_i} q^{\sum_{i=1}^r \binom{A_i+1}{2}} \frac{a_N(q^A, q^{-P-1}) a_M(q^{-A}, q^{-Q})}{(q)_\infty^{m+n}}, \quad (2.4)$$

where  $a_N, a_M$  were defined in formula (1.3) and  $\Delta_{\mu,\nu,\lambda}^{m,n} = \sum_{j=1}^{m-r} M_j Q_j + \sum_{i=1}^{n-r} N_i(P_i + 1)$ .

There are two special cases in which the right hand side takes a simpler form. These two special cases of the theorem were known.

First, if  $m = 0$  then the formula (2.4) reduces to (1.4). More generally if  $r = 0$ , then base of the plane partition decomposes into two connected components and the formula (2.4) become a product  $a_N(q^{-P-1}) a_M(q^{-Q}) / (q)_\infty^{m+n}$ .



**Theorem 3.** *The generating function  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  is equal to the sum*

$$\chi_{\mu,\nu,\lambda}^{n,m}(q) = (-1)^{r(m+n)} \sum_{(\sigma,\tau,A) \in \Theta} (-1)^{|\sigma|+|\tau|+\sum_{i=1}^r A_i} \frac{q^{\Delta^{\sigma,\tau,A}(\mu,\nu,\lambda)}}{(q)_{\infty}^{n+m}}, \quad (2.6)$$

where

$$(\sigma, \tau, A) \in \Theta \Leftrightarrow \sigma \in S_n, \tau \in S_m, A_i = -L_{s_i}, \text{ for } s_1 > \dots > s_r > n - r,$$

and

$$\begin{aligned} \Delta^{\sigma,\tau,A}(\mu, \nu, \lambda) = & \sum_{i=1}^r A_i \left( \frac{A_i + 1}{2} + N_{\sigma(i)} - M_{\tau(i)} \right) \\ & - \sum_{i=r+1}^n (P_{i-r} + 1)(N_{\sigma(i)} - N_i) - \sum_{i=r+1}^m Q_{i-r}(M_{\tau(i)} - M_i). \end{aligned}$$

This theorem will be proven in Section 4. In this proof we consider inequalities  $a_{i,j} \geq a_{i+1,j}$ ,  $a_{i,j} \geq a_{i,j+1}$  and conditions (1.2) as a definition of a polyhedron (infinite dimensional) and the generating function  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  as a sum over integer points in the polyhedron. This sum is calculated using Brion theorem [4]. Each term in (2.6) corresponds to a vertex contribution in Brion theorem.

### 3. LATTICE PATHS

**3.1.** Let  $G$  be an oriented locally finite graph with set of vertices  $V$  and set of edges  $E$ . We also assume that  $G$  have no oriented cycles. For any edge  $e \in E$  we assign a weight  $w(e)$ , for any path  $p = (e_1, e_2, \dots, e_n)$  we define the weight as a product of the edge weights  $w(p) = \prod w(e_i)$ .

For any two vertices  $s, t$  we denote  $P(s \rightarrow t) = \sum_p w(p)$ , where summation goes over all paths from  $s$  to  $t$ . For any sets of  $n$  source vertices  $S = \{s_1, \dots, s_n\}$  and  $n$  target vertices  $T = \{t_1, \dots, t_n\}$  we denote  $P(S \rightarrow T) = \sum_{p_1, \dots, p_n} w(p_1) \cdot \dots \cdot w(p_n)$ , where summation goes over all sets of path such that  $p_i$  goes from  $s_i$  to  $t_i$ . Clearly  $P(S \rightarrow T) = P(s_1 \rightarrow t_1) \cdot \dots \cdot P(s_n \rightarrow t_n)$ .

By  $P_{nc}(S \rightarrow T)$  we denote the sum  $\sum_{p_1, \dots, p_n} w(p_1) \cdot \dots \cdot w(p_n)$  where set of paths is assumed to be without crossings. The Lindström-Gessel-Viennot lemma provides an efficient way to find  $P_{nc}(S \rightarrow T)$ .

**Lemma** (Lindström-Gessel-Viennot; [20],[17]). *For oriented graph  $G$  as above and any sets of sources and targets  $S = \{s_1, \dots, s_n\}, T = \{t_1, \dots, t_n\}$  we have*

$$\sum_{\sigma \in S_n} (-1)^{|\sigma|} P_{nc}(S \rightarrow \sigma(T)) = \det (P(s_i \rightarrow t_j))_{i,j=1}^n$$

In most examples (and in all examples in this paper)  $P_{nc}(S \rightarrow \sigma(T)) \neq 0$  for only one permutation  $\sigma$ . In this case

$$P_{nc}(S \rightarrow \sigma(T)) = (-1)^{|\sigma|} \det (P(s_i \rightarrow t_j))_{i,j=1}^n.$$



**3.2.** In this paper we use graph  $G$  with vertices  $(a + \frac{1}{2}, b)$ , where  $a, b \in \mathbb{Z}, b \geq 0$ . There are two types of edges namely the horizontal ones  $(a + \frac{1}{2}, b) \rightarrow (a + \frac{3}{2}, b)$  ( $\rightarrow$  denotes orientation) and vertical ones  $(a + \frac{1}{2}, b) \rightarrow (a + \frac{1}{2}, b + 1)$  for  $a < 0$  and  $(a + \frac{1}{2}, b) \leftarrow (a + \frac{1}{2}, b + 1)$  for  $a \geq 0$ . The weight of a vertical edge is 1, the weight of a horizontal edge on the line  $y = b$  is  $q^b$ .

Note that the number of paths from  $s = (\frac{1}{2}, b)$  to  $t = (\frac{1}{2} + a, 0)$ ,  $a, b \geq 0$  is equal to binomial coefficient  $\binom{a+b}{b}$ . The number of paths counted with weights is equal to the  $q$ -binomial coefficient  $P(s \rightarrow t) = \left[ \begin{smallmatrix} a+b \\ b \end{smallmatrix} \right]_q$ .

We will use “infinitely remote” source and target vertices, see an example in Fig. 4. We say that the path starts at point  $(-\infty, b)$  if the path contains all sufficiently left edges on the horizontal line  $y = b$ . Similarly we define paths which starts at point  $(a, +\infty)$  or goes to the point  $(+\infty, b)$  or  $(a, +\infty)$ . For example the paths from the point  $s = (-\infty, 0)$  to  $t = (-\frac{1}{2}, +\infty)$  are in one to one correspondence with Young diagrams. And in this case  $P(s \rightarrow t)$  is equal to the generating function of Young diagrams  $1/(q)_\infty$ .

For the ‘infinitely remote’ source and target vertices we need to define the weight of the path. The problem happens for vertices  $(-\infty, b)$  since their paths contain infinitely many horizontal edges on the line  $y = b$  and therefore the weight of these paths are not defined. We divide by  $q^b$  the weight of each horizontal edge (of such paths) over the point  $(i, 0)$ ,  $i < 0$ . Clearly there is no more then one such edge, if there is none we just divide the weight of the path by  $q^b$ . Informally speaking we assign the weight  $q^{-b(\infty/2-1/2)}$  to the vertex  $(-\infty, b)$ . For example for  $s = (-\infty, b)$ ,  $t = (-a - \frac{1}{2}, +\infty)$ ,  $a, b \geq 0$  we have  $P(s \rightarrow t) = q^{-ab}/(q)_\infty$ .

For the  $(+\infty, b)$  we divide by  $q^b$  the weight of each horizontal edge (of path to the  $(+\infty, b)$ ) over the point  $(i, 0)$ ,  $i \geq 0$ . Informally speaking we assign the weight  $q^{-b(\infty/2+1/2)}$  to the vertex  $(-\infty, b)$ . For example for  $s = (a + \frac{1}{2}, +\infty)$ ,  $t = (+\infty, b)$ ,  $a, b \geq 0$  we have  $P(s \rightarrow t) = q^{-(a+1)b}/(q)_\infty$ .

Now we prove the formula (1.4) for the number of plane partitions with  $n$  rows and asymptotic conditions.

**Proposition 3.1.** *Generating function of plane partition  $a_{ij}$ , such that  $1 \leq i \leq n$ ,  $j \in \mathbb{N}$ ,  $(i, j) \notin \lambda$ ,  $\lim_{i \rightarrow \infty} a_{ij} = \nu_i$  has the form*

$$\chi_{\emptyset, \nu, \lambda}^{n, 0}(q) = \frac{q^{\sum_{i=1}^n (\lambda_i + n - i)(\nu_i + n - i)}}{(q)_\infty^n} a_{\nu + \rho_n}(q^{-\lambda - \rho_n}),$$

*Proof.* There is natural bijection between such plane partitions and collections of non intersecting paths from  $S = s_1, \dots, s_n$ ,  $s_i = (\lambda_i + n - i + \frac{1}{2}, +\infty)$  to  $T = t_1, \dots, t_n$ ,  $t_i = (+\infty, \nu_i + n - i)$ . The first row of the plane partition encodes the path from  $s_1$  to  $t_1$ , the second row of the plane partition encodes the path from  $s_2$  to  $t_2$  and so on. The coordinates of the sources and targets are specified in such way that plane partition condition  $a_{i,j} \geq a_{i+1,j}$  is equivalent to the non intersection of paths.

In the Fig. 4 we give an example, where  $n = 3$ ,  $\lambda = (2, 1, 1)$ ,  $\nu = (3, 1, 1)$ .

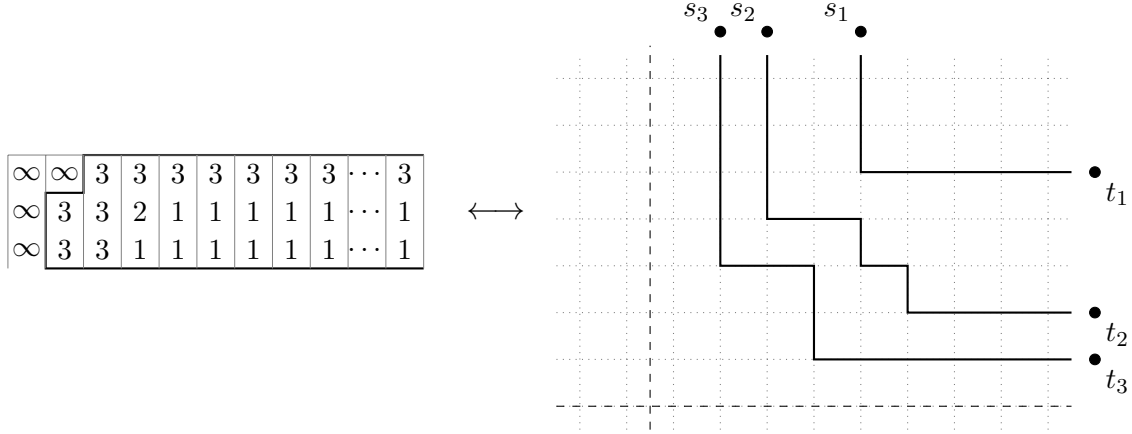


FIGURE 4.

As was noted before we have  $P(s_i \rightarrow t_j) = q^{-(\lambda_i + n - i + 1)(\nu_j + n - j)} / (q)_\infty$ . Therefore using Lindström-Gessel-Viennot lemma we get

$$P(S \rightarrow T) = \det \left( \frac{q^{-(\lambda_i + n - i + 1)(\nu_j + n - j)}}{(q)_\infty} \right).$$

Note that the function  $P(S \rightarrow T)$  differs from  $\chi_{\emptyset, \nu, \lambda}^{n, 0}(q)$  by certain power of  $q$  since on grading on paths differs slightly from the definition (2.1). In particular  $\chi_{\emptyset, \nu, \lambda}^{n, 0}(q)$  has leading term 1 but  $P(S \rightarrow T)$  has leading term  $q^{-\sum_i (\lambda_i + n - i + 1)(\nu_i + n - i)}$ . Multiplying  $P(S \rightarrow T)$  by  $q^{\sum_i (\lambda_i + n - i + 1)(\nu_i + n - i)}$  we get Proposition 3.1.  $\square$

**3.3.** We had not discussed one type of pathes between “infinitely remote” vertices. Namely let  $s = (-\infty, b)$ ,  $t = (+\infty, a)$ . Then the paths from  $s$  to  $t$  are in one to one correspondence with  $V$ -partitions with asymptotic conditions  $\lim_{i \rightarrow \infty} a_i = a$ ,  $\lim_{i \rightarrow \infty} b_i = b$ , see Fig. 5.

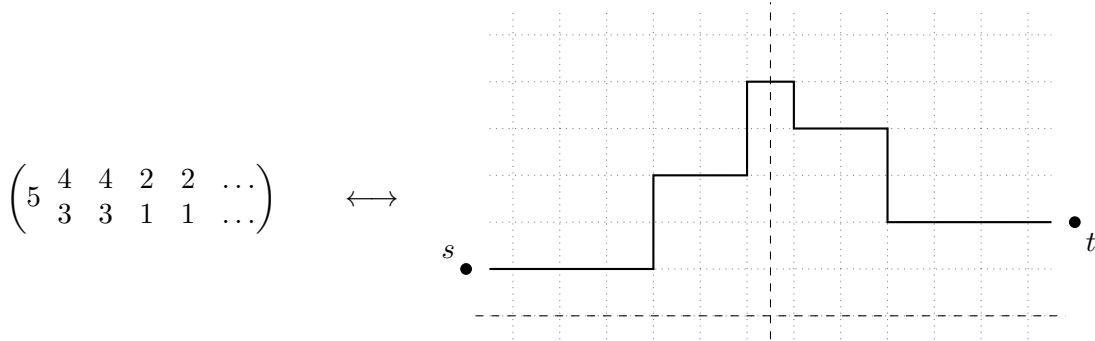


FIGURE 5.

Recall that the generating function of  $V$  partitions was given in Lemma 2.1 and equals  $R(a - b; q)$ . Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* First we use a one to one correspondence between plane partitions satisfying (1.2), (1.1) and certain lattice paths. We decompose the base of plane partition into  $r$  infinite hooks,  $m - r$  infinite columns and  $n - r$  infinite rows.



**3.4.** In this Subsection we prove Lemma 2.2.

*Proof.* We want to calculate the determinant of the matrix

$$M = \begin{pmatrix} \left( \sum_{a \geq 0} (-1)^a q^{\binom{a+1}{2}} q^{(N_j - M_i)a} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} & (q^{-M_i Q_j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-r}} \\ (q^{-N_j(P_i+1)})_{\substack{1 \leq i \leq n-r \\ 1 \leq j \leq n}} & 0 \end{pmatrix}.$$

We decompose this matrix as a product of two (infinite) matrices

$$M = C \begin{pmatrix} 0 & \left( (-1)^a q^{\binom{a+1}{2} - a M_i} \right)_{\substack{1 \leq i \leq m, \\ a \geq 0}} \\ (\delta_{-a-1, P_i})_{\substack{1 \leq i \leq n-r, \\ a < 0}} & 0 \end{pmatrix} \begin{pmatrix} (q^{a N_j})_{\substack{a \in \mathbb{Z}, \\ 1 \leq j \leq n}} & ((-1)^{Q_j} \delta_{a, Q_j})_{\substack{a \in \mathbb{Z}, \\ 1 \leq j \leq m-r}} \end{pmatrix},$$

where  $C = (-1)^{\sum Q_j} q^{-\sum \binom{Q_j+1}{2}}$ . Now we apply Cauchy–Binet formula, the numbers of columns in the minor from the first factor (the numbers of rows in the minor from the second factor) we denote by  $-P_1 - 1, \dots, -P_{n-r} - 1, A_1, \dots, A_r, Q_1, \dots, Q_{m-r}$ , and get

$$\det M = (-1)^{(m-r)(n-r)} \sum_{A_1 > A_2 > \dots > A_r \geq 0} (-1)^{\sum_{i=1}^r A_i} q^{\sum_{i=1}^r \binom{A_i+1}{2}} a_N(q^A, q^{-P-1}) a_M(q^{-A}, q^{-Q})$$

Therefore we proved that (2.3) is equal to (2.4).  $\square$

#### 4. INTEGER POINTS IN POLYHEDRA

**4.1.** In this section we give a combinatorial proof of Theorem 3. We assume that

$$\nu_1 > \dots > \nu_n > \mu_1 > \dots > \mu_m. \quad (4.1)$$

The Theorem 3 valid without this assumption and later, in subsection 4.4 we will explain this.

The proof is based on Brion's theorem which we briefly recall.

Let  $P \subset \mathbb{R}^N$  be a convex polyhedron i.e. an intersection of finite number of half-spaces. Note that  $P$  can be unbounded. For simplicity we assume below that vertices of  $P$  have integer coordinates and edges have rational directions.

For point  $p = (p_1, \dots, p_N) \in \mathbb{Z}^N$  by  $t^p$  we denote  $t_1^{p_1} \dots t_N^{p_N}$ . Define the characteristic function of  $P$  by the formula

$$S(P) = \sum_{p \in P \cap \mathbb{Z}^n} t^p.$$

In this definition  $S(P)$  is a formal series,  $S(P) \in \mathbb{Z}[[t_1^{\pm 1}, \dots, t_N^{\pm 1}]]$ . It can be proven that there exist two Laurent polynomials  $f, g \in \mathbb{Z}[[t_1^{\pm 1}, \dots, t_N^{\pm 1}]]$  such that  $fS(P) = g$ . We denote  $\mathbb{S}(P) = f/g \in \mathbb{Q}(t_1, \dots, t_n)$ . Clearly  $\mathbb{S}(P)$  does not depend on the particular choice of  $f, g \in \mathbb{Z}[[t_1^{\pm 1}, \dots, t_N^{\pm 1}]]$ .

For any vertex  $v \in P$ , we denote by  $\mathcal{K}_v$  its cone i.e. the intersection of half-spaces corresponding to the facets (maximal proper faces) of  $P$  containing  $v$ .

The next theorem is called Brion theorem, standard references for this theorem are [4], [21], [22], [19], for a clear introduction see e.g. [1].

**Theorem** (Brion). *For any convex polyhedron  $P$  with integer vertices and rational directions of edges we have*

$$\mathbb{S}(P) = \sum_v \mathbb{S}(\mathcal{K}_v).$$

Plane partitions satisfying (1.1) and (1.2) are integer points of the polyhedron  $P_{\mu,\nu,\lambda}^{n,m}$  defined as follows

$$P_{\mu,\nu,\lambda}^{n,m}: \quad \begin{cases} t_{i,j} \geq t_{i,j+1} & t_{i,j} \geq \nu_i \geq 0 \\ t_{i,j} \geq t_{i+1,j} & t_{i,j} \geq \mu_j \geq 0, \quad (i,j) \in \mathbb{N}^2 \setminus \lambda. \\ t_{n+1,m+1} = 0 \end{cases} \quad (4.2)$$

Therefore the functions  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  can be computed using Brion theorem.

Two remarks are in order. First, we state Brion theorem for finite dimensional polyhedra but  $P_{\mu,\nu,\lambda}^{n,m}$  is infinite dimensional. Therefore we start from finitization of  $P_{\mu,\nu,\lambda}^{n,m}$ , i.e. for  $H \in \mathbb{N}$  we consider polyhedron  $P_{\mu,\nu,\lambda}^{n,m,(H,H')}$  defined as

$$P_{\mu,\nu,\lambda}^{n,m,(H,H')}: \quad \begin{cases} t_{i,j} \geq t_{i,j+1} & t_{i,H} = \nu_i \geq 0 \\ t_{i,j} \geq t_{i+1,j} & t_{H',j} = \mu_j \geq 0, \quad (i,j) \in \{1, \dots, H'\} \times \{1, \dots, H\} \setminus \lambda. \\ t_{n+1,m+1} = 0 \end{cases} \quad (4.3)$$

Then we take the limit  $H, H' \rightarrow \infty$ .

Second, we need specialization of the function  $\mathbb{S}(P)$  in which  $t_{i,j} \rightarrow q$ . We denote by  $\mathbb{S}_q(P) \in \mathbb{Q}(q)$  the function obtained by composition of  $\mathbb{S}$  and this specialization <sup>2</sup> The limit  $q^{-\Delta(H,H')} \mathbb{S}_q(P_{\mu,\nu,\lambda}^{n,m,(H,H')})$  coincides with  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$ . Here numbers  $\Delta(H,H')$  emerge due to different definitions of grading, see below.

It will be convenient to start from specialization  $t_{i,j} \rightarrow x_{j-i+1}/x_{j-i}$ . We denote by  $\mathbb{S}_x(P) \in \mathbb{Q}(\{x_i\})$  composition of  $\mathbb{S}$  and this specialization. Then we can set  $x_i \rightarrow q^i$  and get  $\mathbb{S}_q(P)$ .

**4.2.** We explain main ideas in the case  $m = 0$ . As the result we get new proof of (1.4).

We start from a description of vertices of the polyhedron  $P_{\emptyset,\nu,\lambda}^{n,0,(H,H')}$ . Since  $t_{n+1,1} = 0$  one can think that indices of coordinates  $t_{i,j}$  satisfy  $1 \leq i \leq n$ ,  $1 \leq j \leq H$ ,  $(i,j) \notin \lambda$ . Any face of our polyhedron is defined by (4.3) where some of inequalities become equalities. For any face we construct graph  $\Gamma$  with vertices  $(i,j) \in (H^n) - \lambda$ . Two vertices  $(i,j)$  and  $(i',j')$  are connected by an edge iff  $t_{i,j} = t_{i',j'}$  for all points of the face and boxes  $(i,j)$  and  $(i',j')$  have a common side.

There exist at least  $n$  connected components in  $\Gamma$  since  $t_{i,H} = \nu_i$  and  $\nu_i > \nu_j$  for  $i > j$  (due to (4.1)). Vertices of our polyhedron are faces of maximal codimension i.e. corresponding to graphs having exactly  $n$  connected components. See an example in Fig. 7.

---

<sup>2</sup>In general, such specialization of a rational function might not be well defined, but in our case  $\mathbb{S}_q(P_{\mu,\nu,\lambda}^{n,m,(H,H')})$  and  $\mathbb{S}_q(\mathcal{K}_v)$ , where  $v$  is vertex of  $P_{\mu,\nu,\lambda}^{n,m,(H)}$  are well defined.

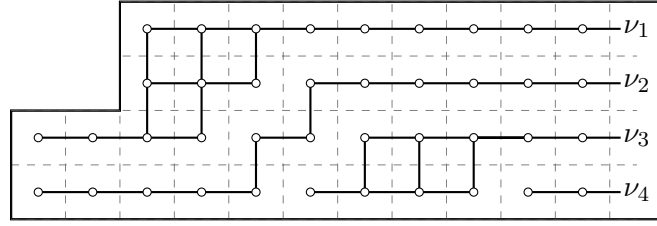


FIGURE 7.

Denote by  $\Gamma_v$  graph corresponding of vertex  $v$ . Denote by  $K_s$  connected components of  $\Gamma_v$ . Each  $K_s$  is a skew Young diagram. Denote by  $\mathcal{K}_{s,v}$  projection of the cone  $\mathcal{K}_v$  on the subspace with coordinated  $t_{i,j}$  for  $(i, j) \in K_i$ . Then we have  $\mathbb{S}(\mathcal{K}_v) = \prod \mathbb{S}(\mathcal{K}_{s,v})$ .

Situation simplifies since for many vertices  $\mathbb{S}_q(\mathcal{K}_v)$  vanishes due to the following result.

**Proposition 4.1** ([23, Theorem 2.1]). *If connected component  $K_s$  has cycles, then  $\mathbb{S}_x(\mathcal{K}_{s,v})$  is equal to 0.*

Therefore by Brion theorem we have

$$\mathbb{S}_q(P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')}) = \sum_v \mathbb{S}_q(\mathcal{K}_v), \quad (4.4)$$

where summation goes over vertices  $v$  such that corresponding graphs  $\Gamma_v$  are acyclic.

Recall that a skew Young diagram  $\alpha - \beta$  is called *ribbon* if it is connected and contains no  $2 \times 2$  block of squares. Due to Proposition 4.1, vertices of  $P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')}$  with nonzero contribution correspond to decompositions of skew diagram  $(H^n) - \lambda$  into  $n$  ribbons such that boxes  $(H, i)$  belong to different ribbons.<sup>3</sup>

The following lemma is standard.

**Lemma 4.1.** *If  $\alpha - \beta$  is a ribbon then there exist  $j, k \in \mathbb{N}$  such that the set  $\{\alpha_i - i\}$  is obtained from the set  $\{\beta_i - i\}$  by replacement of  $\beta_j - j$  by  $\beta_j - j + k$*

The set  $\{\lambda_j - j\}$  is called the set of *shifted parts* of partition  $\lambda$ . Recall useful geometric interpretation of set  $\{j - \lambda_j - \frac{1}{2}\}$  is a set of the coordinates of white balls in the Fig. 3.

The corresponding sets for partitions  $(H^n)$  and  $\lambda$  differs in first  $n$  numbers. Therefore to any decomposition of skew diagram  $(H^n) - \lambda$  into  $n$  ribbons we assign a permutation  $\sigma \in S_n$  such that our ribbons shift  $\lambda_i - i$  to  $H - \sigma(i)$ . Since boxes  $(H, i)$  correspond to different ribbons the order of ribbons is well defined and this assignment is one to one correspondence. For example the graph  $\Gamma_v$  in the Fig. 8 corresponds to the permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ .

We denote by  $v_\sigma$  the acyclic vertex corresponding to  $\sigma \in S_n$ .

**Lemma 4.2.** *For vertex  $v_\sigma$  of polyhedron  $P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')}$  we have*

$$\mathbb{S}_q(\mathcal{K}_{v_\sigma}) = (-1)^{|\sigma|} q^{\Delta^{\sigma, (H)}(\lambda, \nu)} / \prod_{i=1}^n (q)_{H - \sigma(i) - \lambda_i + i - 1}, \quad (4.5)$$

<sup>3</sup>One can compare this to Murnaghan–Nakayama rule.



through the center of the first box in that ribbon then it intersects the ribbon below and so on. Therefore, this line intersects the first box of certain ribbon, say  $K_j$ . Then  $c = \lambda_j - j + 1$  and  $s = \lambda_j - j - \lambda_i + i$ . In such case  $\sigma(i) < \sigma(j)$  since  $K_j$  is below  $K_i$ , but  $i > j$  since  $s > 0$ . And, conversely for any such  $j$  we will have edge  $e_s$  with “−” sign.

Rewriting factors  $1/(1 - q^{-s})$  as  $-q^s/(1 - q^s)$  and using formula for  $h$  we have

$$\mathbb{S}_q(\mathcal{K}_{v_\sigma, i}) = (-1)^{|\{j: j < i, \sigma(j) > \sigma(i)\}|} q^{\nu_{\sigma(i)}(H - \sigma(i) - \lambda_i + i) + \sum_{j < i, \sigma(j) > \sigma(i)} (\lambda_j - j - \lambda_i - i)} \Bigg/ \prod_{s=1}^{H - \sigma(i) - \lambda_i + i - 1} (1 - q^s)$$

Now we can find  $\mathbb{S}_q(\mathcal{K}_{v_\sigma}) = \prod_{i=1}^n \mathbb{S}_q(\mathcal{K}_{v_\sigma, i})$ . Using algebraic identities

$$\sum_{j < i, \sigma(j) > \sigma(i)} (\lambda_j - j - \lambda_i + i) = \sum_{i=1}^n (\lambda_i - i)(\sigma(i) - i) \quad (4.6)$$

and

$$\sum_{i=1}^n \nu_{\sigma(i)}(H - \sigma(i) - \lambda_i + i) + \sum_{i=1}^n (\lambda_i - i)(\sigma(i) - i) = \Delta^{\sigma, (H)}(\lambda, \nu)$$

we get (4.5).  $\square$

Now we can find  $\mathbb{S}_q(P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')})$  using specialization of Brion theorem (4.4)

$$\mathbb{S}_q(P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')}) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} q^{\Delta^{\sigma, (H)}(\lambda, \nu)} / \prod_{i=1}^n (q)_{H - \sigma(i) - \lambda_i + i - 1}.$$

Here we count integer points in  $P_{\emptyset, \nu, \lambda}^{n, 0, (H, H')}$  with the weight  $q^{\sum t_{ij}}$ , which differs from the weight defined in formula (2.1) by  $q^{\Delta^{(H)}}$ , where  $\Delta^{(H)} = \sum_{i=1}^n \nu_i(H - \lambda_i)$ . Using identity

$$\Delta^{\sigma, (H)}(\lambda, \nu) - \Delta^{(H)} = \sum_{i=1}^n (\nu_i + n - i)(\lambda_i + n - i) - \sum_{i=1}^n (\lambda_i + n - i)(\nu_{\sigma(i)} + n - \sigma(i))$$

we see that limit  $\lim_{H \rightarrow \infty} q^{-\Delta^{(H)}} \mathbb{S}_q(P_{\mu, \nu, \lambda}^{n, m, (H, H')})$  coincides with the right side of (1.4)

**4.3.** In this subsection we prove Theorem 3 in the case (4.1). Case of generic  $\nu_i, \mu_j$  will be discussed in subsection 4.4

*Proof.* As in before for any vertex  $v$  we construct graph  $\Gamma_v$ . It follows from Proposition 4.1 that vertices with non zero contribution in  $\mathbb{S}_q$  corresponds to decomposition of skew diagram  $(H^n, m^{H'-n}) - \lambda$  into  $m + n$  ribbons (connected components of  $\Gamma_v$ ), where  $n$  contain boxes  $(i, H)$   $1 \leq i \leq n$  and  $m$  contain boxes  $(H', j)$   $1 \leq j \leq m$ .

For any partition  $\alpha$  we consider the set of shifted parts  $\{\alpha_i - i + n - m + 1\}$ . Note that this set differs from the one used in Lemma 4.1 by  $n - m + 1$ .

We recall notation from Section 2:  $\{L_i = \lambda_i - i + n - m + 1\}$  and  $L_i = P_i + 1$ , for  $1 \leq i \leq n - r$ ,  $L_i \leq 0$  for  $i > n - r$ . The set of shifted parts for  $(H^n, m^{H'-n})$  equals  $\{H + n - m - 1, \dots, H - m + 1, 0, \dots, -H' + n + 1, -H' + n - m, \dots\}$ . Due to Lemma 4.1 addition of  $n$  ribbons containing boxes  $(i, H)$  should replace  $n$  numbers



$B_1 > B_2 > \dots > B_n$ ,  $B_i \in \{L_s | s \in \mathbb{N}\}$  by numbers  $H + n - m - i$ . The numbers  $P_i + 1$  should belong to the set  $\{B_i\}$ , therefore

$$(B_1, B_2, \dots, B_n) = (P_1 + 1, \dots, P_{n-r} + 1, -A_r, \dots, -A_1), \quad (4.7)$$

where  $A_i \in \{-L_s | s > n - r\}$ . For any vertex we assign permutation  $\sigma \in S_n$  such that our  $n$  ribbons replace  $B_i$  by  $H + n - m + 1 - \sigma(i)$ . These data  $\sigma \in S_n$  and  $A_i \in \{-L_s | s > n - r\}$  encodes  $n$  ribbons containing  $(i, H)$ .

Due to Lemma 2.3 the set of shifted parts  $\{L_i\}$  have  $m - r$  negative holes (missing negative numbers) in integers  $-Q_j$ . After adding first  $n$  ribbons we have  $m$  holes in integers  $-C_j$  where

$$(C_1, C_2, \dots, C_m) = (Q_1, \dots, Q_{m-r}, A_1, \dots, A_r). \quad (4.8)$$

The set of shifted parts for  $(H^n, m^{H'-n})$  have holes in integers  $-H' + n + j$ ,  $1 \leq j \leq m$ . Therefore the  $m$  ribbons containing boxes  $(H, j)$  replace integers  $-H' + n + \tau(j)$  by  $-C_j$ .

We denote by  $v_{\sigma, \tau, A}$  corresponding vertex, it is easy to see that these data determine vertex uniquely. In the example in Fig. 10 we have  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ ,  $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ ,  $A_1 = 2, A_2 = 0$ .

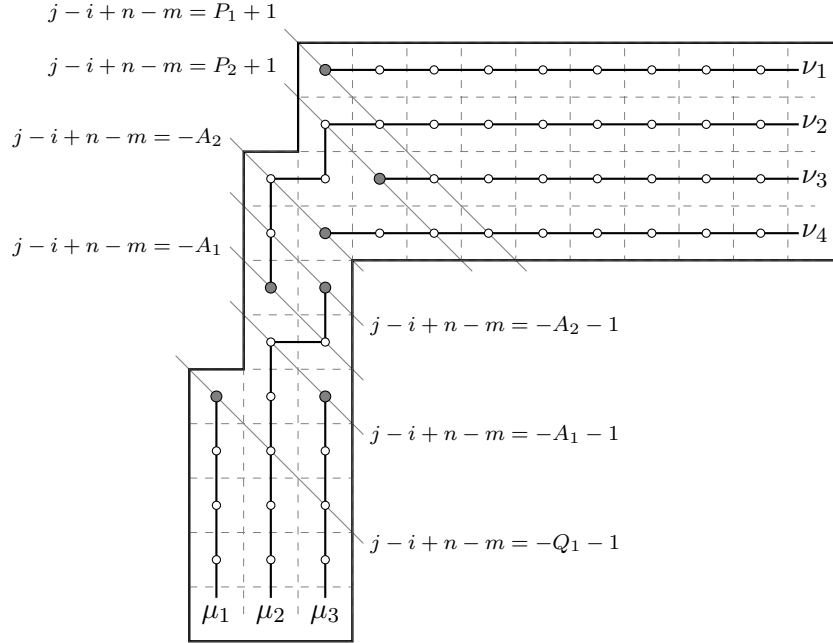


FIGURE 10.

Therefore by Brion theorem we have

$$\mathbb{S}_q(P_{\mu, \nu, \lambda}^{n, m, (H, H')}) = \sum_{\sigma, \tau, A} \mathbb{S}_q(\mathcal{K}_{v_{\sigma, \tau, A}}), \quad (4.9)$$

where  $\sigma \in S_n, \tau \in S_m, A_i = -L_{s_i}$ , for  $s_1 > \dots > s_r > n - r$  and  $A_i < H' - n$ .

**Lemma 4.3.** *For vertex  $v_{\sigma,\tau,A}$  of polyhedron  $P_{\mu,\nu,\lambda}^{n,m,(H)}$  we have*

$$\mathbb{S}_q(\mathcal{K}_{v_{\sigma,\tau,A}}) = \frac{(-1)^{|\sigma|+|\tau|+\sum A_i-i+1} q^{\Delta^{\sigma,\tau,A,(H)}(\mu,\nu,\lambda)}}{\prod_{i=1}^n (q)^{H-\sigma(i)+n-m-B_i} \prod_{j=1}^m (q)^{H'-\tau(j)+m-n-C_j-1}}, \quad (4.10)$$

where

$$\begin{aligned} \Delta^{\sigma,\tau,A,(H)}(\mu,\nu,\lambda) = & \sum_{i=1}^r A_i \left( \frac{A_i+1}{2} - N_{\sigma(n-i+1)} + M_{\tau(m-r+i)} \right) + \sum_{i=1}^{n-r} (P_i+1)(-N_{\sigma(i)}+n-i) + \\ & + \sum_{j=1}^{m-r} Q_j(-M_{\tau(j)}+m-j) + \sum_{i=1}^n \nu_i(H+n-m-i+1) + \sum_{j=1}^m \mu_j(H'+n-m-j) \end{aligned} \quad (4.11)$$

and  $N_i = \nu_i + n - i$ ,  $M_j = \mu_j + m - j$ .

*Proof.* The proof of this lemma is analogous to the one of Lemma 4.2. In the denominator we have product  $(q)_{h-1}$  where  $h$  is the length of the ribbon. The power of  $q$  in the numerator made from two summands. The first one

$$\sum_{i=1}^n \nu_{\sigma(i)}(H - \sigma(i) + (n - m) + 1 - B_i) + \sum_{j=1}^m \mu_{\tau(j)}(H' - \tau(j) + m - n - C_j)$$

is the sum of weights of vertices  $v_i$  in cones  $\mathcal{K}_{v_{\sigma,\tau,A},i}$  corresponding to ribbons. The second one

$$\sum_{i=1}^r \binom{A_i+1}{2} + \sum_{i=1}^r A_i(2i - r - 1) + \sum_{i=1}^n B_i(\sigma(i) - i) + \sum_{j=1}^m C_j(\tau(j) - j)$$

comes from rewriting  $1/(1-q^{-s})$  as  $-q^s/(1-q^s)$ . Here we also used identity (4.6). Putting all things together and using (4.7), (4.8) we get (4.11).  $\square$

Now we find  $\chi_{\mu,\nu,\lambda}^{n,m}(q)$  as the limit  $\lim_{H \rightarrow \infty} q^{-\Delta^{(H,H')}} \mathbb{S}_q(P_{\mu,\nu,\lambda}^{n,m,(H,H')})$ , where

$$\begin{aligned} \Delta^{(H,H')} = & \sum_{i=1}^{n-r} \nu_i(H - P_i - i + n - m) + \sum_{j=1}^{m-r} \mu_j(H' - Q_j - j + m - n) \\ & + \sum_{i=n-r+1}^n \nu_i(H - i + n - m + 1) + \sum_{j=m-r+1}^m \mu_j(H' - j + m - n). \end{aligned}$$

It is easy to see that  $\Delta^{\sigma,\tau,A,(H,H')}(\mu,\nu,\lambda) - \Delta^{(H,H')} = \Delta^{\tilde{\sigma},\tilde{\tau},A}(\mu,\nu,\lambda)$  for

$$\tilde{\sigma} = \sigma \circ \begin{pmatrix} 1 & \dots & r & r+1 & \dots & n \\ n & \dots & n-r+1 & 1 & \dots & n-r \end{pmatrix}, \quad \tilde{\tau} = \tau \circ \begin{pmatrix} 1 & \dots & r & r+1 & \dots & m \\ m-r+1 & \dots & m & 1 & \dots & m-r \end{pmatrix}$$

and we get formula (2.6).  $\square$

**4.4.** In the previous subsection we proved Theorem 3 under the assumption (4.1)

$$\nu_1 > \dots > \nu_n > \mu_1 > \dots > \mu_m.$$

But this theorem holds for any  $\nu, \mu$  since it is equivalent to Theorem 1 proven in section 4. In this subsection we explain how to get rid of condition (4.1) in context of Brion theorem.

First note that if some inequalities between  $\nu_i, \mu_j$  become equalities then the polyhedron  $P_{\mu, \nu, \lambda}^{n, m, (H, H')}$  degenerates. This degeneration changes combinatorial structure, in particular some of the vertices merge. But one can ignore this when using Brion theorem (see arguments in [13, Sec. 8]). Therefore the Theorem 3 still holds.

Now fix any strong order  $\mathfrak{o}$  on  $\nu_i, \mu_j$  such that  $\nu_{i_1} > \nu_{i_2}, \nu_{j_1} > \nu_{j_2}$  for  $i_1 < i_2$  and  $j_1 < j_2$ . Proposition 4.1 implies that vertices with non zero contribution to  $\mathbb{S}_q(P_{\mu, \nu, \lambda}^{n, m, (H, H')})$  correspond to decompositions of the skew diagram  $(H^n, m^{H'-n}) - \lambda$  into  $m + n$  ribbons (of which  $n$  contain boxes  $(i, H)$   $1 \leq i \leq n$  and  $m$  contain boxes  $(H', j)$   $1 \leq j \leq m$ ) such that the layout of ribbons is compatible with the order  $\mathfrak{o}$ .

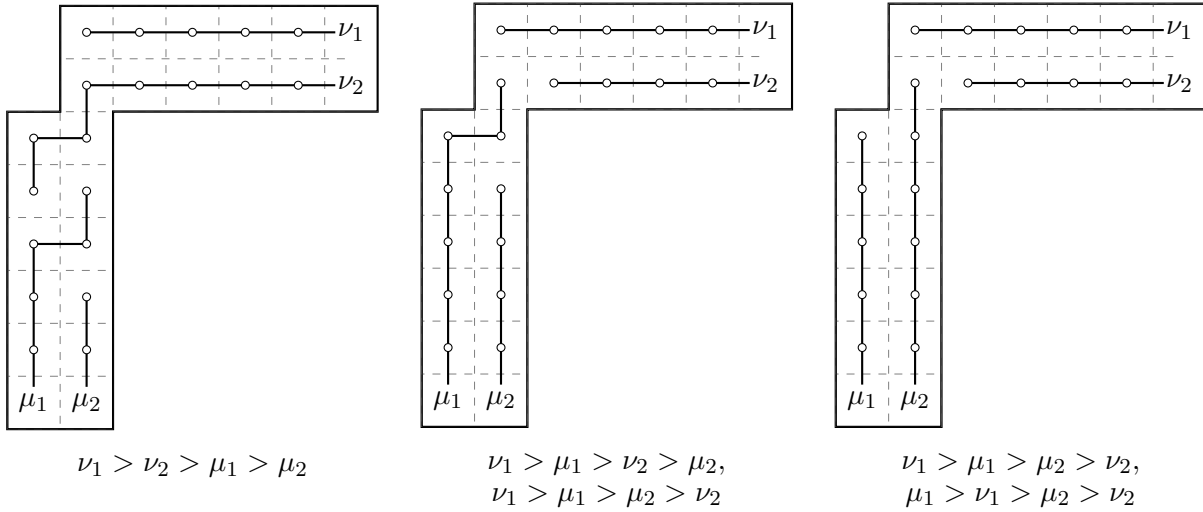


FIGURE 11. Some examples of decompositions and compatible orders

The set of vertices depends on the order  $\mathfrak{o}$ . But the contribution of a vertex is a product of ribbon contributions, and each such contribution is defined for any  $\nu_i$  and  $\mu_j$  (not necessarily satisfying inequalities from  $\mathfrak{o}$ ). So for any given order  $\mathfrak{o}$  the function  $\mathbb{S}_q(P_{\mu, \nu, \lambda}^{n, m, (H, H')})(q)$  computed using Brion theorem is defined for any nonnegative integer numbers  $\nu_i, \mu_j$  not necessarily satisfying  $\mathfrak{o}$ . For any given  $n, m, \mu, \nu, \lambda$  we denote this function just by  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$  and its limit as  $H, H' \rightarrow \infty$  as  $\mathbb{S}_{\mathfrak{o}}(q)$ .

**Example 4.1.** Let  $n = m = 1, \lambda = \emptyset$ . We have two possible orders  $\mathfrak{o}: \nu_1 > \mu_1$  and  $\mathfrak{o}': \mu_1 > \nu_1$ . For  $\mathfrak{o}$  we have  $H' - 1$  vertices and using Brion Theorem we obtain

$$\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q) = \sum_{a=0}^{H'-2} \frac{(-1)^a q^{\binom{a+1}{2}} q^{(H+a)\nu_1 + (H'-a-1)\mu_1}}{(q)_{H+a-1} (q)_{H'-a-2}}.$$

For the order  $\mathfrak{o}'$  we have  $H - 1$  vertices and using Brion Theorem we obtain

$$\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q) = \sum_{b=0}^{H-2} \frac{(-1)^b q^{\binom{b+1}{2}} q^{(H-b-1)\nu_1 + (H'+b)\mu_1}}{(q)_{H-b-2} (q)_{H'+b-1}},$$

Using binomial theorem in form

$$\sum_{c=0}^N \frac{(-1)^c q^{\binom{c}{2}} x^c}{(q)_c (q)_{N-c}} = \frac{1}{(q)_n} \prod_{i=1}^n (1 - xq^{i-1})$$

for  $x = q^{\nu_1 - \mu_1 + H - 2}$ ,  $N = H + H' - 3$  and obvious observation that right side vanishes for  $x = q^c$ ,  $0 \leq c < N$  we have

$$\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q) = \mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q), \text{ for } 0 \leq \nu_1 - \mu_1 + H - 2 < H + H' - 3.$$

It is convenient to rewrite the last inequality as  $1 - H' \leq \nu_1 - \mu_1 < H - 1$ . Therefore for any given  $\nu_1, \mu_1$  and large enough  $H, H'$  functions computed for different orders coincide and in the limit  $\mathbb{S}_{\mathfrak{o}}(q) = \mathbb{S}_{\mathfrak{o}'}(q)$ .

**Proposition 4.2.** *If  $n - H' \leq \nu_i - \mu_j < H - m$  for any  $i, j$  then for any two orders  $\mathfrak{o}, \mathfrak{o}'$  we have*

$$\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q) = \mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q).$$

*Proof.* It is enough to consider the case when  $\mathfrak{o}, \mathfrak{o}'$  differ only by an elementary transposition of  $\nu_i$  and  $\mu_j$  (in  $\mathfrak{o}$ :  $\nu_i > \mu_j$  and  $\mathfrak{o}'$ :  $\mu_j > \nu_i$ ). Recall that summands in  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$  and  $\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q)$  correspond to decompositions of skew diagram  $(H^n, m^{H'-n}) - \lambda$  into  $m + n$  ribbons. There are three possibilities.

*Case 1.* Ribbons containing boxes  $(H, i)$  and  $(j, H')$  have no common edges. Corresponding summands appear both in  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$  and  $\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q)$ .

*Case 2.* Union of ribbons containing boxes  $(H, i)$  and  $(j, H')$  is a ribbon. Denote this ribbon by  $\alpha - \beta$ . Fix ribbons containing other end-boxes  $(H, i')$  and  $(j', H')$  for  $i' \neq i$  and  $j' \neq j$ . For such summands  $\alpha - \beta$  is divided by an internal edge  $e$  into two ribbons containing  $(H, i)$  and  $(j, H')$ . Denote by  $\mathbb{S}_{e, \alpha - \beta}(q)$  the product of contributions of these two ribbons. If the edge  $e$  is horizontal then the corresponding term appears in  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$ , otherwise in  $\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q)$ .

**Lemma 4.4.** *Suppose  $\alpha - \beta$  is a skew ribbon such that  $\alpha - \beta$  lies in rectangle  $H \times H'$  and contains boxes  $(H, 1)$  and  $(1, H')$ . If  $1 - H' \leq \nu_1 - \mu_1 < H - 1$  then*

$$\sum_{e: \text{horizontal}} \mathbb{S}_{e, \alpha - \beta}(q) = \sum_{e: \text{vertical}} \mathbb{S}_{e, \alpha - \beta}(q)$$

This lemma is a generalization of Example 4.1 and by straightforward calculation reduces to  $q$ -binomial theorem. Due to this lemma in this case the contributions to  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$  and  $\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q)$  are equal to each other.

*Case 3.* Union of ribbons containing boxes  $(H, i)$  and  $(j, H')$  is a connected skew Young diagram  $\alpha - \beta$  but not a ribbon. Informally it means that  $\alpha - \beta$  has width 2 in the middle. In this case there is two ways to decompose  $\alpha - \beta$  into two ribbons. Corresponding two terms are equal to each other, and one goes to  $\mathbb{S}_{\mathfrak{o}}^{(H, H')}(q)$  and other to  $\mathbb{S}_{\mathfrak{o}'}^{(H, H')}(q)$ .  $\square$

**Remark 4.1.** The calculation in Case 3 is essentially the last step in the proof of [23, Theorem 2.1] (see our Proposition 4.1).

Tending  $H, H' \rightarrow \infty$  we get from Proposition 4.2 that the function  $\mathbb{S}_{\mathfrak{o}}(q)$  does not depend on the order  $\mathfrak{o}$ . For actual order of  $\nu_i, \mu_j$  this function coincides with  $\chi_{\mu, \nu, \lambda}^{n, m}(q)$  and for order (4.1) this function coincides with right side of (2.6). Hence we proved Theorem 3 for any nonnegative integers  $\nu_i, \mu_j$ .

## 5. ALGEBRAS, REPRESENTATIONS AND RESOLUTIONS

**5.1.** For the reference of quantum toroidal algebra  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  one can use [10, Sec 2] but our notation slightly differs from loc. cit.<sup>4</sup>

Fix complex numbers  $\epsilon_i$ , where  $i = 1, 2, 3$  and should be viewed as a mod 3 residues. We assume that  $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ . Denote  $q_i = e^{\epsilon_i}$ ,  $\vec{q} = (q_1, q_2, q_3)$ . We assume further that  $q_1, q_2, q_3$  are generic, i.e. for integers  $l, m, n \in \mathbb{Z}$ ,  $q_1^l q_2^m q_3^n = 1$  holds only if  $l = m = n$ . We set

$$g(z, w) = \prod_{i=1}^3 (z - q_i w), \quad \kappa_r = \prod_{i=1}^3 (q_i^{r/2} - q_i^{-r/2}) = \sum_{i=1}^3 (q_i^r - q_i^{-r}), \quad \delta(z) = \sum_{m \in \mathbb{Z}} z^m.$$

The algebra  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  is generated by  $E_m, F_m, H_r$  where  $m, r \in \mathbb{Z}$ ,  $r \neq 0$  and invertible central elements  $K_0, C, C^\perp$ . In order to write relations we form the currents (generating functions of operators)

$$E(z) = \sum_{m \in \mathbb{Z}} E_m z^{-m}, \quad F(z) = \sum_{m \in \mathbb{Z}} F_m z^{-m}, \quad K^\pm(z) = K_0(C^\perp)^{\pm 1} \exp \left( \sum_{r > 0} \mp \frac{\kappa_r}{r} H_{\pm r} z^{\mp r} \right).$$

The relations have form

$$\begin{aligned} g(z, w)E(z)E(w) + g(w, z)E(w)E(z) &= 0, & g(w, z)F(z)F(w) + g(z, w)F(w)F(z) &= 0, \\ K^\pm(z)K^\pm(w) &= K^\pm(w)K^\pm(z), & \frac{g(C^{-1}z, w)}{g(Cz, w)}K^-(z)K^+(w) &= \frac{g(w, C^{-1}z)}{g(w, Cz)}K^+(w)K^-(z), \\ g(z, w)K^\pm(C^{(-1 \mp 1)/2}z)E(w) + g(w, z)E(w)K^\pm(C^{(-1 \mp 1)/2}z) &= 0, \\ g(w, z)K^\pm(C^{(-1 \pm 1)/2}z)F(w) + g(z, w)F(w)K^\pm(C^{(-1 \pm 1)/2}z) &= 0, \\ [E(z), F(w)] &= \frac{1}{\kappa_1} \left( \delta\left(\frac{Cw}{z}\right)K^+(w) - \delta\left(\frac{Cz}{w}\right)K^-(z) \right), \\ \text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [E(z_1), [E(z_2), E(z_3)]] &= 0, & \text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [F(z_1), [F(z_2), F(z_3)]] &= 0. \end{aligned}$$

There exists an action of the group  $SL(2, \mathbb{Z})$  on the toroidal algebra  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  by automorphisms. We denote by  $E_m^\perp, F_m^\perp, H_r^\perp$  images of generators  $E_m, F_m, H_r$  after rotation  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

Denote by  $d$  the grading operator

$$[d, E_m] = mE_m, \quad [d, F_m] = mF_m, \quad [d, H_r] = dH_r, \quad [d, C] = [D, C^\perp] = [d, K_0] = 0.$$

<sup>4</sup> currently there is no standard convention to the notations, even for the algebra besides  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  other names  $\mathcal{E}$ ,  $\mathcal{E}_1$ ,  $\mathbf{SH}$ ,  $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$  are also used in the literature.

Sometimes it is convenient to consider  $d$  as an additional generator  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ . Let  $V$  be a representation of  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  such that one can define action of  $d$  on the space  $V$  with finite dimensional eigenspaces. By the character  $\chi(V)$  denote the trace of operator  $D = q^{-d}$  where  $q$  is a formal variable.

The algebra  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  has the following formal coproduct<sup>5</sup>

$$\begin{aligned}\Delta(H_r) &= H_r \otimes 1 + C^{-n} \otimes H_r, \quad \Delta(H_{-r}) = H_{-r} \otimes C^r + 1 \otimes H_{-r}, \quad r > 0 \\ \Delta(E(z)) &= E(C_2^{-1}z) \otimes K^+(C_2^{-1}z) + 1 \otimes E(z), \\ \Delta(F(z)) &= F(z) \otimes 1 + K^-(C_1^{-1}z) \otimes F(C_1^{-1}z), \\ \Delta(X) &= X \otimes X, \quad \text{for } X = K_0, C, C^\perp, D,\end{aligned}\tag{5.1}$$

where  $C_1 = C \otimes 1$ ,  $C_2 = 1 \otimes C$ .

In all representations of  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$  considered in this paper we have  $C^\perp = K_0 = 1$ .

In the paper [9] authors defined the MacMahon modules of the algebra  $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ . The MacMahon modules depend on the parameters  $v, c$  and three partitions  $\nu, \mu, \lambda$ , where  $c$  is a value of central element  $C$ . These modules are denoted by  $\mathcal{M}_{\mu, \nu, \lambda}(v, c)$ . This module has the basis  $|a\rangle$ , where  $a$  is a plane partition which satisfy condition (1.2). The action of  $d$  on  $\mathcal{M}_{\mu, \nu, \lambda}(v, c)$  is defined by  $d|a\rangle = |a||a\rangle$ . Therefore the character  $\chi(\mathcal{M}_{\mu, \nu, \lambda}(v, c))$  is equal to the generating function of plane partitions satisfying (1.2).

The modules  $\mathcal{M}_{\lambda, \mu, \nu}(v)$  were originally defined by the explicit formulas for the action of “rotated” generators  $E_m^\perp, F_m^\perp, H_r^\perp$  in the basis labeled by plane partitions. For example the action of  $K^{\perp, \pm}(z)$  have the form

$$K^{\perp, \pm}(z)|a\rangle = \frac{1 - c^2 v/z}{1 - v/z} \prod_{(i, j, k) \in a} \psi_{i, j, k}(v/z)|a\rangle\tag{5.2}$$

where

$$\psi_{i, j, k}(v/z) = \frac{(1 - q_1^{i-1} q_2^j q_3^k v/z)(1 - q_1^i q_2^{j-1} q_3^k v/z)(1 - q_1^i q_2^j q_3^{k-1} v/z)}{(1 - q_1^{i+1} q_2^j q_3^k v/z)(1 - q_1^i q_2^{j+1} q_3^k v/z)(1 - q_1^i q_2^j q_3^{k+1} v/z)}.$$

Notation  $(i, j, k) \in a$  means that  $(i, j, k)$  belongs to the corresponding 3d Young diagram, see Fig. 1. It is easy to see that the product in right side of (5.2) after cancellation of common factors becomes finite. The highest weight of  $\mathcal{M}_{\mu, \nu, \lambda}(v, c)$  is given by the formula (5.2) applied for “minimal” plane partition  $a$  satisfying conditions (1.2).

For the generic values  $c, v, q_1, q_2, q_3$  the module  $\mathcal{M}_{\mu, \nu, \lambda}(v, c)$  is irreducible. But for  $c = q_1^{n/2} q_2^{m/2}$  (and generic  $v, q_1, q_2, q_3$ ) this module has one singular vector. The quotient by the submodule generated by this vector is irreducible. This quotient is denoted by  $\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v)$  and has the basis  $|a\rangle$  where  $a$  is a plane partition, satisfying both conditions (1.2) and (1.1). Recall that partitions  $\lambda, \mu, \nu$  satisfy  $l(\nu) \leq n$ ,  $l(\mu) \leq m$ , and  $\lambda_{n+1} < m + 1$ .

Therefore the character  $\chi(\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v))$  is equal to the generating function  $\chi_{\mu, \nu, \lambda}^{n, m}(q)$  defined in the introduction. This is the representation theoretic interpretation of the left side of

<sup>5</sup>note that our  $E_m, F_m, H_r$  are called  $e_m^\perp, f_m^\perp, h_r^\perp$  in [10] (up to rescaling of  $h_r$ )

(2.3), (2.4), (2.6). Now we will discuss the representation theoretic interpretation of the right side.

**5.2.** It is difficult to write down the explicit action of generators  $E_n, F_n, H_m$  in modules  $\mathcal{N}_{\lambda, \mu, \nu}^{n, m}(v)$ . Now we recall construction of another class of modules, namely the Fock modules and intertwining operators between them, which are called screening operators. Then we sketch construction of MacMahon modules  $\mathcal{N}_{\lambda, \mu, \nu}^{n, m}(v)$  in these terms.

The name of Fock modules over  $U_{\check{q}}(\check{\mathfrak{gl}}_1)$  comes from the fact that the representation space is identified with the Fock module over some Heisenberg algebra. In these representations the currents  $E(z), F(z), K^{\pm}(z)$  are given in terms of the Heisenberg algebra (as combination of vertex operators).

We start from the basic Fock modules  $\mathcal{F}_u^{(i)}$ , where  $u = e^p$ ,  $p \in \mathbb{C}$  and  $i = 1, 2, 3$ . The central charges of these representations are  $C = q_i^{1/2}$ ,  $K_0 = C^{\perp} = 1$ . This representation space is a module over Heisenberg algebra with generators  $a_n$ ,  $n \in \mathbb{Z}$  and relations

$$[a_r, a_s] = r \frac{(q_i^{r/2} - q_i^{-r/2})^3}{-\kappa_r} \delta_{r+s, 0}. \quad (5.3)$$

We denote by  $v_u^{(i)}$  the highest weight vector of  $\mathcal{F}_u^{(i)}$  such that

$$a_r v_u^{(i)} = 0, \text{ for } r > 0; \quad a_0 v_u^{(i)} = -\frac{\epsilon_i^2 p}{\epsilon_1 \epsilon_2 \epsilon_3} v_u^{(i)}.$$

Then the representation  $\rho_u^{(i)}$  in the space  $\mathcal{F}_u^{(i)}$  is defined by the formulae

$$\begin{aligned} \rho_u^{(i)}(E(z)) &= \frac{u(1 - q_i)}{\kappa_1} \exp \left( \sum_{r=1}^{\infty} \frac{q_i^{-r/2} \kappa_r}{r(q_i^{r/2} - q_i^{-r/2})^2} a_{-r} z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{\kappa_r}{r(q_i^{r/2} - q_i^{-r/2})^2} a_r z^{-r} \right), \\ \rho_u^{(i)}(F(z)) &= \frac{u^{-1}(1 - q_i^{-1})}{\kappa_1} \exp \left( \sum_{r=1}^{\infty} \frac{-\kappa_r}{r(q_i^{r/2} - q_i^{-r/2})^2} a_{-r} z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{-q_i^{r/2} \kappa_r}{r(q_i^{r/2} - q_i^{-r/2})^2} a_r z^{-r} \right), \\ \rho_u^{(i)}(H_r) &= \frac{a_r}{q_i^{r/2} - q_i^{-r/2}}, \quad \rho_u^{(i)}(K_0) = \rho_u^{(i)}(C^{\perp}) = 1, \quad \rho_u^{(i)}(C) = q_i^{1/2}, \\ \rho_u^{(i)}(d) v_u^{(i)} &= \Delta_i(p) v_u^{(i)}, \quad \Delta_i(p) = \frac{(p + \epsilon_i)^3 - p^3}{6\epsilon_1 \epsilon_2 \epsilon_3}. \end{aligned} \quad (5.4)$$

Note that generally speaking the operators  $a_0$  and  $d$  can act on the highest weight vector  $v_u^{(i)}$  by any number. Our choice is convenient for the formulas below, for example the screening operators will commute with  $d$  due to our choice.

We introduce operators  $\widehat{Q}$  by the relation  $[a_n, \widehat{Q}] = \frac{-\epsilon_i^3}{\epsilon_1 \epsilon_2 \epsilon_3} \delta_{n, 0}$ . The exponent  $e^{x\widehat{Q}}$  acts from  $\mathcal{F}_u^{(i)}$  to  $\mathcal{F}_{q_i^x u}^{(i)}$  and maps highest weight vector  $v_u^{(i)}$  to the vector  $v_{q_i^x u}^{(i)}$ .

The modules  $\mathcal{F}_u^{(i)}$  are irreducible. In terms of rotated generators its highest weight has the form

$$K^{\perp, \pm}(z) v_u^{(i)} = \frac{1 - q_i u/z}{1 - u/z} v_u^{(i)} \quad (5.5)$$

In particular the highest weight of Fock module  $\mathcal{F}_u^{(1)}$  coincides with the highest weight of MacMahon modules  $\mathcal{M}_{\emptyset, \{\nu_1\}, \{\lambda_1\}}(v)$  for  $c = q_1^{1/2}$  and  $u = v q_2^{\nu_1} q_3^{\lambda_1}$  (see (5.2)). Therefore

the irreducible quotient  $\mathcal{N}_{\emptyset, \{\nu_1\}, \{\lambda_1\}}^{1,0}(v)$  is isomorphic to the Fock module  $\mathcal{F}_u^{(1)}$ . Similarly the MacMahon module  $\mathcal{N}_{\{\mu_1\}, \emptyset, \{\lambda_1\}}^{0,1}(v)$  is isomorphic to the Fock module  $\mathcal{F}_u^{(2)}$ , where  $u = vq_1^{\mu_1}q_3^{\lambda_1}$ .

The highest weight of the module  $\mathcal{N}_{\mu, \nu, \lambda}^{n,n}(v)$  given by the rational function which can be decomposed as the product of several factors of the type (5.5). Therefore  $\mathcal{N}_{\mu, \nu, \lambda}^{n,n}(v)$  is isomorphic to subquotient of Fock module, which can be defined as tensor product of basic ones

$$\mathcal{F}_{u_1}^{(i_1)} \otimes \mathcal{F}_{u_2}^{(i_2)} \otimes \dots \otimes \mathcal{F}_{u_k}^{(i_k)}.$$

Now we will describe image of  $U_{\vec{q}}(\mathfrak{gl}_1)$  in these representations.

**5.3.** First we consider tensor product  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$ . This tensor product was essentially elaborated in paper [12] which we follow. We introduce the Heisenberg generators  $h_n$  acting on  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$

$$h_{-r} = q_i^{-r}(a_{-r} \otimes 1) - q_i^{-r/2}(1 \otimes a_{-r}), \quad h_r = q_i^{r/2}(a_r \otimes 1) - q_i^r(1 \otimes a_r), \quad r > 0. \quad (5.6)$$

These operators satisfy  $[h_n, \Delta(H_m)] = 0$  and this condition determines them up to normalization. In our normalization we have

$$[h_r, h_s] = r \frac{(q_i^r - q_i^{-r})(q_i^{r/2} - q_i^{-r/2})^2}{-\kappa_r} \delta_{r+s,0}.$$

Denote  $\widehat{Q}_1 = \widehat{Q} \otimes 1$ ,  $\widehat{Q}_2 = 1 \otimes \widehat{Q}$ ,  $u_1 = e^{p_1}$ ,  $u_2 = e^{p_2}$ . Following [8] we introduce two screening currents

$$\begin{aligned} S_+^{ii}(z) &= e^{\frac{\epsilon_{i+1}}{\epsilon_i}(\widehat{Q}_1 - \widehat{Q}_2)} z^{\frac{p_2 - p_1 + \epsilon_i}{\epsilon_{i+1}}} \exp \left( \sum_{r=1}^{\infty} \frac{-(q_{i+1}^{r/2} - q_{i+1}^{-r/2})}{r(q_i^{r/2} - q_i^{-r/2})} h_{-r} z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{(q_{i+1}^{r/2} - q_{i+1}^{-r/2})}{r(q_i^{r/2} - q_i^{-r/2})} h_r z^{-r} \right) \\ S_-^{ii}(z) &= e^{\frac{\epsilon_{i-1}}{\epsilon_i}(\widehat{Q}_1 - \widehat{Q}_2)} z^{\frac{p_2 - p_1 + \epsilon_i}{\epsilon_{i+1}}} \exp \left( \sum_{r=1}^{\infty} \frac{-(q_{i-1}^{r/2} - q_{i-1}^{-r/2})}{r(q_i^{r/2} - q_i^{-r/2})} h_{-r} z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{(q_{i-1}^{r/2} - q_{i-1}^{-r/2})}{r(q_i^{r/2} - q_i^{-r/2})} h_r z^{-r} \right). \end{aligned} \quad (5.7)$$

**Lemma 5.1.** *The image of  $U_{\vec{q}}(\mathfrak{gl}_1)$  (including the operator  $d$ ) commutes with the following screening operators*

$$S_+^{ii} = \oint S_+^{ii}(z) dz, \quad S_-^{ii} = \oint S_-^{ii}(z) dz. \quad (5.8)$$

This lemma follows from a direct computation.

The algebra which commute with the screening operators are usually called [8] the  $q$ -deformed  $W$ -algebra. The commutativity with screening operators (5.12) determine  $W$ -algebra, which is called  $q$ -deformed  $W$ -algebra of  $\mathfrak{gl}_2$ . It can be proved that the image of  $U_{\vec{q}}(\mathfrak{gl}_1)$  in the representation  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$  coincide with the  $q$ -deformed  $W$ -algebra of  $\mathfrak{gl}_2$ , or  $W_{\vec{q}}(\mathfrak{gl}_2)$  for short.

Note that the screening currents formally commute

$$S_+^{ii}(z) S_-^{ii}(w) = \frac{1}{(z - q_i^{-1/2} w)(z - q_i^{1/2} w)} :S_+^{ii}(z) S_-^{ii}(w): = S_+^{ii}(w) S_-^{ii}(z).$$



For generic  $u_1, u_2$  the module  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$  is irreducible. But if  $u_2 = u_1 q_{i+1}^s q_{i-1}^t$  where  $s, t \in \mathbb{Z}$  and  $st > 0$  then it is not so. If  $s, t > 0$ , then  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$  has irreducible factor isomorphic to MacMahon module  $\mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v)$  (one can see this from comparison of highest weights). Moreover, this MacMahon module can be written as a cohomology of complex consisting of Fock modules.

The simplest example of such complex given if  $s$  or  $t$  equal to 1

$$\begin{aligned} 0 \rightarrow \mathcal{F}_{vq_2^{\nu_1-1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1-s}}^{(1)} &\xrightarrow{S_+^{11}} \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_1-1}q_3^{\lambda_1-s}}^{(1)} \rightarrow \mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_1-s+1\}}^{2,0}(v) \rightarrow 0 \\ 0 \rightarrow \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1-1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_1-s}q_3^{\lambda_1}}^{(1)} &\xrightarrow{S_+^{11}} \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_1-s}q_3^{\lambda_1-1}}^{(1)} \rightarrow \mathcal{N}_{\emptyset, \{\nu_1, \nu_1-s+1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v) \rightarrow 0 \end{aligned}$$

For generic module  $\mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v)$  the corresponding short exact sequence have the form

$$\begin{aligned} 0 \rightarrow \mathcal{F}_{vq_2^{\nu_2-1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_2-1}}^{(1)} &\xrightarrow{(S_+^{11})^{\nu_1-\nu_2+1}} \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_2-1}q_3^{\lambda_2-1}}^{(1)} \rightarrow \mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v) \rightarrow 0 \\ 0 \rightarrow \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_2-1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_2-1}q_3^{\lambda_1}}^{(1)} &\xrightarrow{(S_-^{11})^{\lambda_1-\lambda_2+1}} \mathcal{F}_{vq_2^{\nu_1}q_3^{\lambda_1}}^{(1)} \otimes \mathcal{F}_{vq_2^{\nu_2-1}q_3^{\lambda_2-1}}^{(1)} \rightarrow \mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v) \rightarrow 0 \end{aligned} \quad (5.9)$$

where operators  $(S_{\pm}^{11})^r$  should be considered as a  $r$ -fold integral over the appropriate cycle with the appropriate additional factor. Now we compute the Euler characteristic of (5.9). Using  $\chi(\mathcal{F}_{u_1}^{(1)} \otimes \mathcal{F}_{u_2}^{(1)}) = q^{\Delta_1(p_1) + \Delta_1(p_2)} / (q)_{\infty}^2$ , where  $\Delta_i(p)$  is defined in (5.4) we have

$$\chi\left(\mathcal{N}_{\emptyset, \{\nu_1, \nu_1\}, \{\lambda_1, \lambda_2\}}^{2,0}(v)\right) = q^{\Delta} \frac{q^{-(\lambda_1+1)(\nu_1+1)-\lambda_2\nu_2} - q^{-(\lambda_1+1)\nu_1-(\lambda_1+1)(\nu_2+1)}}{(q)_{\infty}^2},$$

where

$$\Delta = -\Delta_1(p + \nu_1\epsilon_2 + \lambda_1\epsilon_3) - \Delta_1(p + (\nu_2 - 1)\epsilon_2 + (\lambda_2 - 1)\epsilon_3) + (\lambda_1 + 1)(\nu_1 + 1) + \lambda_2\nu_2,$$

and  $e^p = v$ . Up to factor  $q^{\Delta}$  this formula coincide with (2.6) (or with its special case (1.4)).

In a similar manner one can construct resolutions of  $\mathcal{N}_{\{\mu_1, \mu_2\}, \emptyset, \{\lambda_1, \lambda_2\}}^{0,2}(v)$  in terms of  $\mathcal{F}_{u_1}^{(2)} \otimes \mathcal{F}_{u_2}^{(2)}$ .

Below we will discuss the algebra of screening operators which commute with image of algebra  $U_{\bar{q}}(\mathfrak{gl}_1)$  in the representation  $\mathcal{F}_{u_1}^{(1)} \otimes \dots \mathcal{F}_{u_n}^{(1)}$ . This system of screening operators coincides with one studied in [8], the algebra which commutes with them (i.e. image of  $U_{\bar{q}}(\mathfrak{gl}_1)$ ) is  $W_{\bar{q}}(\mathfrak{gl}_n)$ . We claim that one can construct resolution of  $\mathcal{N}_{\emptyset, \nu, \lambda}^{n,o}(v)$  in terms of the modules  $\mathcal{F}_{u_1}^{(1)} \otimes \dots \otimes \mathcal{F}_{u_n}^{(1)}$ . See Section 5.5 for more details.

**5.4.** Second we consider the tensor product  $\mathcal{F}_{u_1}^{(1)} \otimes \mathcal{F}_{u_2}^{(2)}$ . We introduce the Heisenberg generators  $h_n$  acting on  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$

$$\begin{aligned} h_{-r} &= \frac{q_1^{-r}(q_2^{r/2} - q_2^{-r/2})}{q_1^{r/2} - q_1^{-r/2}}(a_{-r} \otimes 1) - \frac{q_1^{-r/2}(q_1^{r/2} - q_1^{-r/2})}{q_2^{r/2} - q_2^{-r/2}}(1 \otimes a_{-r}), \\ h_r &= \frac{q_1^{r/2}(q_2^{r/2} - q_2^{-r/2})}{q_1^{r/2} - q_1^{-r/2}}(a_r \otimes 1) - \frac{q_3^{-r/2}(q_1^{r/2} - q_1^{-r/2})}{q_2^{r/2} - q_2^{-r/2}}(1 \otimes a_r), \end{aligned} \quad r > 0. \quad (5.10)$$

These operators satisfy  $[h_n, \Delta(H_m)] = 0$  and this condition determines them up to normalization. In our normalization we have

$$[h_r, h_s] = r\delta_{r+s,0}.$$

Similar to previous case, denote  $\widehat{Q}_1 = \widehat{Q} \otimes 1$ ,  $\widehat{Q}_2 = 1 \otimes \widehat{Q}$ ,  $u_1 = e^{p_1}$ ,  $u_2 = e^{p_2}$  and introduce screening current

$$S^{12}(z) = e^{\frac{\epsilon_2}{\epsilon_1}\widehat{Q}_1 - \frac{\epsilon_1}{\epsilon_2}\widehat{Q}_2} z^{\frac{p_2 - p_1 + \epsilon_2}{\epsilon_3}} \exp\left(\sum_{r=1}^{\infty} \frac{1}{-r} h_{-r} z^r\right) \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} h_r z^{-r}\right). \quad (5.11)$$

**Lemma 5.2.** *The image of  $U_{\vec{q}}(\dot{\mathfrak{gl}}_1)$  (including the operator  $d$ ) commutes with the following screening operator*

$$S^{12} = \oint S^{12}(z) dz. \quad (5.12)$$

This lemma follows from direct computation. Note that in this case we have only one screening current contrary to two currents  $S_-^{ii}(z), S_+^{ii}(z)$  above. Also note that the commutation relations of  $S^{12}(z)$  do not depend on  $q_1, q_2, q_3$ .

$$S^{12}(z)S^{12}(w) = (z - w) :S^{12}(z)S^{12}(w): = -S^{12}(w)S^{12}(z),$$

i.e. we have a fermion screening current. In particular we have  $S^{12}S^{12} = 0$ . We will call the algebra which commute with the operator  $S^{12}$  by  $q$ -deformed  $W$ -algebra of  $\dot{\mathfrak{gl}}_{1|1}$  or  $W_{\vec{q}}(\dot{\mathfrak{gl}}_{1|1})$  for short. The arguments for such name will be given below.

For generic  $u_1, u_2$  the module  $\mathcal{F}_{u_1}^{(i)} \otimes \mathcal{F}_{u_2}^{(i)}$  is irreducible. But in the resonance case we have a nontrivial intertwining operators between such modules and can construct complex with cohomology  $\mathcal{N}_{\{\mu_1\}, \{\nu_1\}, \emptyset}^{1,1}(v)$ .

Namely, for  $\nu_1 \geq \mu_1$  we have exact sequence

$$\dots \xrightarrow{S^{12}} \mathcal{F}_{vq_2^{-2}q_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_1^3q_3^{\mu_1}}^{(2)} \xrightarrow{S^{12}} \mathcal{F}_{vq_2^{-1}q_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_1^2q_3^{\mu_1}}^{(2)} \xrightarrow{S^{12}} \mathcal{F}_{vq_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_1q_3^{\mu_1}}^{(2)} \rightarrow \mathcal{N}_{\{\mu_1\}, \{\nu_1\}, \emptyset}^{1,1}(v) \rightarrow 0, \quad (5.13)$$

and for  $\nu_1 \leq \mu_1$  we have exact sequence

$$0 \rightarrow \mathcal{N}_{\{\mu_1\}, \{\nu_1\}, \emptyset}^{1,1}(v) \xrightarrow{S^{12}} \mathcal{F}_{vq_2q_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_3^{\mu_1}}^{(2)} \xrightarrow{S^{12}} \mathcal{F}_{vq_2^2q_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_1^{-1}q_3^{\mu_1}}^{(2)} \xrightarrow{S^{12}} \mathcal{F}_{vq_2^3q_3^{\nu_1}}^{(1)} \otimes \mathcal{F}_{vq_1^{-2}q_3^{\mu_1}}^{(2)} \xrightarrow{S^{12}} \dots \quad (5.14)$$

(Note that for  $\nu_1 = \mu_1$  both complexes exist). Taking the euler characteristics we get

$$\chi\left(\mathcal{N}_{\{\mu_1\}, \{\nu_1\}, \emptyset}^{1,1}(v)\right) = q^{\Delta} R(\nu_1 - \mu_1, q) \quad (5.15)$$

where  $e^p = v$  and  $\Delta = -\Delta_1(p + \nu_1\epsilon_3) - \Delta_2(p + \epsilon_1 + \mu_1\epsilon_3)$ . Up to factor  $q^{\Delta}$  this formula coincide with (2.6) for this special case.

**5.5.** Now we want to consider tensor products

$$\mathcal{F}_{u_1}^{(1)} \otimes \dots \mathcal{F}_{u_n}^{(1)} \otimes \mathcal{F}_{u_{n+1}}^{(2)} \otimes \dots \otimes \mathcal{F}_{u_{n+m}}^{(2)}. \quad (5.16)$$

Similar to previous discussion we expect that the MacMahon module  $\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v)$  is given as cohomology of resolutions consisting of modules of the type (5.16). For example one can easily see that the central charges of  $\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v)$  and (5.16) are equal to  $c = q_1^{n/2} q_2^{m/2}$ .

Moreover we can consider another ordering in tensor product (5.16). For generic parameters  $u$  any tensor product of  $n$  modules  $\mathcal{F}^{(1)}$  and  $m$  modules  $\mathcal{F}^{(2)}$  is isomorphic to the product in order (5.16). For each pair of neighbor Fock modules  $\mathcal{F}_{u_l}^{(i_l)} \otimes \mathcal{F}_{u_{l+1}}^{(i_{l+1})}$  we constructed before the screening operators  $\left(S_*^{i_l i_{l+1}}\right)_{l,l+1}$ , where indices  $l, l+1$  label Fock modules in which this operator act and  $*$  =  $\pm$  if  $i_l \neq i_{l+1}$ . For any  $l, *$  the operator  $\left(S_*^{i_l i_{l+1}}\right)_{l,l+1}$  commutes with the image  $U_{\vec{q}}(\mathfrak{gl}_1)$ .

For modules (5.16) it is convenient to decompose the corresponding screening operators on 3 systems:

$$\begin{aligned} \mathfrak{S}_1 &= \left\{ (S_-^{11})_{i,i+1} \mid 0 < i < n \right\}, \\ \mathfrak{S}_2 &= \left\{ (S_+^{22})_{j,j+1} \mid n < j < m+n \right\}, \\ \mathfrak{S}_3 &= \left\{ (S_+^{11})_{i,i+1}, (S_-^{12})_{n,n+1}, (S_-^{22})_{j,j+1} \mid 0 < i < n, n < j < n+m \right\}. \end{aligned} \quad (5.17)$$

We will denote by  $W_{\vec{q}}(\mathfrak{gl}_{n|m})$  the algebra which commutes with this system of screening operators. Different orderings in (5.16) correspond to different Borel subalgebras in  $\mathfrak{gl}_{n|m}$ .

Now discuss representation theoretic interpretation of character formulas (2.6), (2.4), (2.3).

- Each term in the sum of right side of (2.6) has the form of  $q^\Delta / (q_\infty)^{n+m}$ . This is the character of the Fock module (5.16). Therefore it is natural to expect that right side of (2.6) is an Euler characteristic of a resolution, which consist of Fock modules (5.16). The terms in this resolution should be labeled by  $(\sigma, \tau, A) \in \Theta$  as in (2.6). We will say that this resolution is a materialization of a formula (2.6).

This resolution is a generalization of resolutions (5.9) and (5.13), (5.14) discussed above. The intertwining operators in this resolution could be constructed using screening operators. The construction of such resolution is unknown (actually we didn't give proof of the existence of (5.9) and (5.13), (5.14) but these particular cases are rather easy).

Even in the case  $m = 0$  which correspond to  $W_{\vec{q}}(\mathfrak{gl}_n)$  we didn't find this resolution in the literature. It is well known that conformal limit of such resolution exists. For the construction of intertwining operators in the  $W_{\vec{q}}(\mathfrak{gl}_n)$  case see [11].

- There exist another class of representations of  $W_{\vec{q}}(\mathfrak{gl}_{n|m})$ . The algebra which commute with screening operators  $\left\{ (S_\pm^{11})_{i,i+1}, (S_\pm^{22})_{j,j+1} \mid 0 < i < n, n < j < n+m \right\}$  (the same set as before except  $(S^{12})_{n,n+1}$ ) is isomorphic to the product  $W_{\vec{q}}(\mathfrak{gl}_n) \otimes W_{\vec{q}}(\mathfrak{gl}_m)$ . Therefore algebra  $W_{\vec{q}}(\mathfrak{gl}_{n|m})$  is a subalgebra of  $W_{\vec{q}}(\mathfrak{gl}_n) \otimes W_{\vec{q}}(\mathfrak{gl}_m)$ . The last algebra has representations of the form  $\mathcal{N}_{\emptyset, \nu, \nu'}^{n,0}(v_1) \otimes \mathcal{N}_{\mu, \emptyset, \mu'}^{0,m}(v_2)$ . Since the character of each factor is given by formula (1.4) we have

$$\chi(\mathcal{N}_{\emptyset, \nu, \nu'}^{n,0}(v_1) \otimes \mathcal{N}_{\mu, \emptyset, \mu'}^{0,m}(v_2)) = q^\Delta \frac{a_{\nu+\rho_n}(q^{-\nu'-\rho_n}) a_{\mu+\rho_n}(q^{-\mu'-\rho_n})}{(q)_\infty^{n+m}},$$

for certain  $\Delta$ . The right side of character formula (2.4) is a linear combination of such terms. Therefore it is natural to expect that there exists a resolution consisting of modules

$\mathcal{N}_{\emptyset, \nu, \nu'}^{n,0}(v_1) \otimes \mathcal{N}_{\mu, \emptyset, \mu'}^{0,m}(v_2)$  with the cohomology  $\mathcal{N}_{\mu, \nu, \lambda}^{n,m}(v)$ . This resolutions should be a materialization of character formula (2.4). See also Section 5.7 below.

- Consider tensor product

$$\mathcal{F}_{u_1}^{(1)} \otimes \dots \mathcal{F}_{u_{n-r}}^{(1)} \otimes \mathcal{F}_{u_{n-r+1}}^{(2)} \otimes \dots \otimes \mathcal{F}_{u_{n+m-2r}}^{(2)} \otimes \left( \mathcal{F}_{u_{n+m-2r}}^{(1)} \otimes \mathcal{F}_{u_{n+m-2r+2}}^{(2)} \otimes \dots \mathcal{F}_{u_{n+m-1}}^{(1)} \otimes \mathcal{F}_{u_{n+m}}^{(2)} \right). \quad (5.18)$$

As we mention before this product is isomorphic to (5.16).

Consider the following system of screening operators

$$\left\{ (S^{12})_{l,l+1} \mid n+m-2r < l < n+m, l \equiv n+m+1 \pmod{2} \right\}.$$

The  $W$ -algebra, which commutes with this system is  $W_{\vec{q}}(\dot{\mathfrak{gl}}_1)^{\otimes(n+m-2r)} \otimes W_{\vec{q}}(\dot{\mathfrak{gl}}_{1|1})^{\otimes r}$ , where  $W_{\vec{q}}(\dot{\mathfrak{gl}}_1)^{\otimes(n+m-2r)}$  is just Heisenberg algebra acting on first  $n+m-2r$  factors on (5.18). But this system is a subsystem of all screening operators acting (5.18), therefore  $W_{\vec{q}}(\dot{\mathfrak{gl}}_{n|m})$  is a subalgebra of  $W_{\vec{q}}(\dot{\mathfrak{gl}}_1)^{\otimes(n+m-2r)} \otimes W_{\vec{q}}(\dot{\mathfrak{gl}}_{1|1})^{\otimes r}$ . The last  $W$ -algebra has the representations in the space

$$\mathcal{F}_{u_1}^{(1)} \otimes \dots \mathcal{F}_{u_{n-r}}^{(1)} \otimes \mathcal{F}_{u_{n-r+1}}^{(2)} \otimes \dots \otimes \mathcal{F}_{u_{n+m-2r}}^{(2)} \otimes \mathcal{N}_{\{m_1\}, \{n_1\}, \emptyset}^{1,1}(v_1) \otimes \dots \otimes \mathcal{N}_{\{m_r\}, \{n_r\}, \emptyset}^{1,1}(v_r). \quad (5.19)$$

Due to (5.15) the character of this representations equals  $\prod_{i=1}^r R(d_i; q) \cdot q^\Delta / (q_\infty)^{m+n-2r}$ , where  $d_i = n_i - m_i$  and certain  $\Delta$ .

If we compute determinant in right side of (2.3) we get the linear combination of terms  $\prod_{i=1}^r R(d_i; q) \cdot q^\Delta / (q_\infty)^{m+n-2r}$ . Therefore it is natural to conjecture the existence of the resolution of  $\mathcal{N}_{\mu, \nu, \lambda}^{n,m}(v)$  which consists of modules of the type (5.19). And this resolution should be a materialization of the character formula (2.3).

**5.6.** Now we consider conformal limit of the previous construction. We rescale  $\epsilon_i \rightarrow \hbar \epsilon_i$ ,  $p_l \rightarrow \hbar p_l$  and then send  $\hbar$  to zero i.e. send all parameters  $q_i, u_l$  to 1 with the certain speed. The limit of screening operators introduced above is well defined.

Denote by  $\bar{a}_{n,l}$  the limit of generators  $a_n$  acting on the  $l$ -th factor in (5.16). As the limit of (5.3) we get

$$[\bar{a}_{r,i}, \bar{a}_{s,i}] = -r \frac{\epsilon_1^2}{\epsilon_2 \epsilon_3} \delta_{r+s,0}, \quad [\bar{a}_{r,j}, \bar{a}_{s,j}] = -r \frac{\epsilon_2^2}{\epsilon_1 \epsilon_3} \delta_{r+s,0}, \quad 1 \leq i \leq n, \quad n+1 \leq j \leq n+m,$$

and  $[\bar{a}_{r,l}, \bar{a}_{s,l'}] = 0$ , for  $l \neq l'$ . In other words operators  $\bar{a}_{n,l}$  forms the Heisenberg algebra constructed from space  $\mathbb{C}^{n+m}$  with the scalar product  $(\cdot, \cdot)$  given in orthogonal basis  $e_l$  by formula:

$$(e_i, e_i) = -\frac{\epsilon_1^2}{\epsilon_2 \epsilon_3}, \quad (e_j, e_j) = -\frac{\epsilon_2^2}{\epsilon_1 \epsilon_3}, \quad 1 \leq i \leq n, \quad n+1 \leq j \leq n+m.$$

It is convenient to introduce

$$\varphi_l(z) = \sum_{r \in \mathbb{Z} \setminus 0} \frac{\bar{a}_{r,l}}{r} z^{-r} + \bar{a}_{0,l} \log z + \bar{Q}_l, \quad (5.20)$$

where  $\bar{Q}_l$  is a limit  $\widehat{Q}_l$ , with relation  $[a_{0,l}, \bar{Q}_l] = (e_l, e_l)$ .

The limit of screening currents have the form

$$\begin{aligned}\lim_{h \rightarrow 0} (S_{\pm}^{11}(z))_{i,i+1} &= \exp\left(\sum_{l=1}^{n+m} (\alpha_{\pm,i})_l \varphi_l(z)\right), \quad 1 \leq i \leq n, \\ \lim_{h \rightarrow 0} (S^{12}(z))_{n,n+1} &= \exp\left(\sum_{l=1}^{n+m} (\alpha_n)_l \varphi_l(z)\right) \\ \lim_{h \rightarrow 0} (S_{\pm}^{22}(z))_{j,j+1} &= \exp\left(\sum_{l=1}^{n+m} (\alpha_{\pm,j})_l \varphi_l(z)\right), \quad n+1 \leq j \leq n+m.\end{aligned}$$

We consider the vectors  $\alpha_{\pm,i}, \alpha_n, \alpha_{\pm,j}$  as the vectors in  $\mathbb{C}^{n+m}$  and they have the form

$$\begin{aligned}\alpha_{+,i} &= \frac{\epsilon_2}{\epsilon_1} e_i - \frac{\epsilon_2}{\epsilon_1} e_{i+1}, & \alpha_n &= \frac{\epsilon_2}{\epsilon_1} e_n - \frac{\epsilon_1}{\epsilon_2} e_{n+1}, & \alpha_{-,j} &= \frac{\epsilon_1}{\epsilon_2} e_j - \frac{\epsilon_1}{\epsilon_2} e_{j+1} \\ \alpha_{-,i} &= \frac{\epsilon_3}{\epsilon_1} e_i - \frac{\epsilon_3}{\epsilon_1} e_{i+1}, & & & \alpha_{+,j} &= \frac{\epsilon_3}{\epsilon_2} e_j - \frac{\epsilon_3}{\epsilon_2} e_{j+1}.\end{aligned}$$

Slightly abusing notation we will say that  $\alpha \in \mathfrak{S}_I$ , for  $I = 1, 2, 3$  if the corresponding screening operator belongs the  $\mathfrak{S}_I$ .

For any  $\beta, \gamma \in \mathbb{C}^{n+m}$  the commutation relations of vertex operators have the form

$$\exp\left(\sum_{l=1}^{n+m} \beta_l \varphi_l(z)\right) \exp\left(\sum_{l=1}^{n+m} \gamma_l \varphi_l(w)\right) = (z-w)^{(\beta, \gamma)} \exp\left(\sum_{l=1}^{n+m} \beta_l \varphi_l(z) + \gamma_l \varphi_l(w)\right)$$

In particular, if  $(\beta, \gamma) \in 2\mathbb{Z}$  then the corresponding vertex operators formally commute and if  $(\beta, \gamma) \in 2\mathbb{Z} + 1$  then the corresponding vertex operators formally anticommute. It is easy to see that if two vectors  $\alpha, \alpha'$  belong to different systems  $\mathfrak{S}$  then the scalar product  $(\alpha, \alpha') \in \mathbb{Z}$ . And the Gramian matrices for vectors from  $\mathfrak{S}$  are given below

$$\begin{aligned}\mathfrak{S}_1: & \begin{pmatrix} \frac{-2\epsilon_3}{\epsilon_2} & \frac{\epsilon_3}{\epsilon_2} & 0 & \dots & 0 \\ \frac{\epsilon_3}{\epsilon_2} & \frac{-2\epsilon_3}{\epsilon_2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{-2\epsilon_3}{\epsilon_2} & \frac{\epsilon_3}{\epsilon_2} \\ 0 & \dots & 0 & \frac{\epsilon_3}{\epsilon_2} & \frac{-2\epsilon_3}{\epsilon_2} \end{pmatrix}, & \mathfrak{S}_2: & \begin{pmatrix} \frac{-2\epsilon_3}{\epsilon_1} & \frac{\epsilon_3}{\epsilon_1} & 0 & \dots & 0 \\ \frac{\epsilon_3}{\epsilon_1} & \frac{-2\epsilon_3}{\epsilon_1} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{-2\epsilon_3}{\epsilon_1} & \frac{\epsilon_3}{\epsilon_1} \\ 0 & \dots & 0 & \frac{\epsilon_3}{\epsilon_1} & \frac{-2\epsilon_3}{\epsilon_1} \end{pmatrix}, \\ \mathfrak{S}_3: & \begin{pmatrix} \frac{-2\epsilon_2}{\epsilon_3} & \frac{\epsilon_2}{\epsilon_3} & 0 & \dots & 0 & \dots & \dots & \dots & 0 \\ \frac{\epsilon_2}{\epsilon_3} & \frac{-2\epsilon_2}{\epsilon_3} & \ddots & \ddots & \vdots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & \ddots & \ddots & \frac{-2\epsilon_2}{\epsilon_3} & \frac{\epsilon_2}{\epsilon_3} & \ddots & & & \vdots \\ 0 & \dots & 0 & \frac{\epsilon_2}{\epsilon_3} & 1 & \frac{\epsilon_1}{\epsilon_3} & 0 & \dots & 0 \\ \vdots & & & \ddots & \frac{\epsilon_1}{\epsilon_3} & \frac{-2\epsilon_1}{\epsilon_3} & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \vdots & \ddots & \ddots & \frac{-2\epsilon_1}{\epsilon_3} & \frac{\epsilon_1}{\epsilon_3} \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 & \frac{\epsilon_1}{\epsilon_3} & \frac{-2\epsilon_1}{\epsilon_3} \end{pmatrix}.\end{aligned}$$

The Gramian matrix corresponding to  $\mathfrak{S}_1$  is equal to the Cartan matrix of  $\mathfrak{sl}_n$  multiplied by  $-\frac{\epsilon_3}{\epsilon_2}$ . The  $W$ -algebra commuting with screening operators with such Gramian matrix is called  $W(\mathfrak{sl}_n)$  [6]. In our case the  $W$  algebra is  $W(\mathfrak{gl}_n) \otimes \text{Heis}^{\otimes m}$  since there exists  $m+1$  dimensional Heiseberg algebra which commutes with all screening operators from  $\mathfrak{S}_1$ . Similarly, commutativity with screening operators from  $\mathfrak{S}_2$  determines  $W$  algebra  $\text{Heis}^{\otimes n} \otimes W(\mathfrak{gl}_m)$ .

The Gramian matrix corresponding to  $\mathfrak{S}_3$  have blocks corresponding to  $\mathfrak{sl}_n$ ,  $\mathfrak{sl}_m$  and fermionic screening operator between them. Therefore we call the  $W$ -algebra commuting with this system  $W(\mathfrak{gl}_{n|m})$ . The  $W(\mathfrak{gl}_{n|1})$  case was considered in [16], but we didn't find any reference for general  $n, m$ . Note that our  $W$ -algebras differ from ones introduced in [18].

**5.7.** Standard statement in the theory of vertex algebras is an equivalence of the abelian categories of certain representations of vertex algebra and certain representations of quantum group. This is a statement similar to Drinfeld–Kohno or Kazhdan–Lusztig theorem. In fact this equivalence of categories is an equivalence of braided tensor categories but we do not need tensor structure here.

It is known that certain category of representations of vertex algebra  $W(\mathfrak{gl}_n)$  is equivalent to certain category of representations of quantum group  $U_q \mathfrak{gl}_n \otimes U_{q'} \mathfrak{gl}_n$ , where parameters  $q, q'$  are given in terms of  $\epsilon_1, \epsilon_2$  (people also use modular double of  $U_q \mathfrak{gl}_n$ ). We conjecture that the same relation holds for vertex algebra  $W(\mathfrak{gl}_{n|m})$  and quantum group  $U_q \mathfrak{gl}_{n|m} \otimes U_{q'} \mathfrak{gl}_n \otimes U_{q''} \mathfrak{gl}_m$  for certain  $q, q', q''$  given in terms of  $\epsilon_1, \epsilon_2$ .

Denote by  $L_\nu^{(n)}$  the finite dimensional irreducible representation of  $U_{q'}(\mathfrak{gl}_n)$ , recall that these representations are labeled by partitions  $\nu$  such that  $l(\nu) \leq n$ . Similarly denote by  $L_\mu^{(m)}$  the finite dimensional irreducible representation of  $U_{q''} \mathfrak{gl}_n$ . For  $U_q \mathfrak{gl}_{n|m}$  consider only tensor irreducible representations i.e. irreducible submodules of the tensor powers of  $\mathbb{C}^{n|m}$ . They are labeled by partitions  $\lambda$  such that  $\lambda_{n+1} < m+1$  [2], [26] and we denote them by  $L_\lambda^{(n|m)}$ . There are also Kac modules for  $U_q \mathfrak{gl}_{n|m}$  which are induced from the representation of parabolic subalgebra  $\mathfrak{p}$  on the tensor product representation of  $L_\nu^{(n)} \otimes L_{\mu'}^{(m)}$   $U_q \mathfrak{gl}_n \otimes U_q \mathfrak{gl}_m \subset \mathfrak{p} \subset U_q \mathfrak{gl}_{n|m}$ . We denote such modules by  $V_{\nu', \mu'}$ .

We conjecture that under equivalence above the tensor product of irreducible modules  $L_\lambda^{(n|m)} \otimes L_\nu^{(n)} \otimes L_\mu^{(m)}$  goes to the conformal limit of  $\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v)$ . And tensor product  $V_{\nu', \mu'} \otimes L_\nu^{(n)} \otimes L_\mu^{(m)}$  goes to the conformal limit of  $\mathcal{N}_{\emptyset, \nu, \nu'}^{n, 0}(v_1) \otimes \mathcal{N}_{\mu, \emptyset, \mu'}^{0, m}(v_2)$ .

In paper [5] Cheng, Kwon and Lam constructed a resolution in terms of Kac modules of the tensor module of  $\mathfrak{gl}_{n|m}$ . Taking the conjectural  $q$ -deformation of this resolution we have a complex which consist of modules  $V_{\nu', \mu'}$  with the cohomology  $L_\lambda^{(n|m)}$ . Multiplying by  $L_\nu^{(n)} \otimes L_\mu^{(m)}$  and applying equivalence we get the resolution of the conformal limit of  $\mathcal{N}_{\mu, \nu, \lambda}^{n, m}(v)$  in terms of conformal limit of  $\mathcal{N}_{\emptyset, \nu, \nu'}^{n, 0}(v_1) \otimes \mathcal{N}_{\mu, \emptyset, \mu'}^{0, m}(v_2)$ . This resolution should be a materialization of (2.4), its  $q$ -deformation was discussed above in Section 5.5.

The Euler characteristic of resolution constructed in [5] yields the following formula

$$s_\lambda(x|y) = \sum_{\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r \geq r-m} (-1)^{\sum \alpha_i} s_{\pi+m-r, -\alpha}(x) s_{\alpha, \kappa}(y) \prod_{1 \leq i \leq n, 1 \leq j \leq m} \left(1 + \frac{y_j}{x_i}\right).$$

Here notations  $\pi$ ,  $\kappa$  were introduced in Section 2.1,  $s_\mu(x)$  is a Schur polynomial i.e. character of  $L_\mu^{(m)}$  and  $s_\lambda(x|y)$  is a hook Schur polynomial (or super-Schur polynomial) i.e. character of  $L_\lambda^{(n|m)}$ . This formula resembles our character formula (2.4).

**Remark 5.1.** Moens and van der Jeugt found another formula for the character of  $L_\lambda^{(n|m)}$

$$s_\lambda(x|y) = \frac{(-1)^{mn-r} \prod_{i,j} \left(1 + \frac{y_j}{x_i}\right)}{V(x_1, \dots, x_n) V(y_1, \dots, y_m)} \det \begin{pmatrix} \left( \sum_{a \geq 0} (-1)^a x_j^{-a-1+m} y_i^a \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} & (y_i^{Q_j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-r}} \\ (x_j^{P_i+m})_{\substack{1 \leq i \leq n-r \\ 1 \leq j \leq n}} & 0 \end{pmatrix}. \quad (5.21)$$

This formula is similar to our formula (2.3). It is natural to conjecture that there is a resolution which is a materialization of (5.21) and under the equivalence this resolution goes to resolution which is materialization of (2.3).

## REFERENCES

- [1] M. Beck, C. Haase, F. Sottile, *Formulas of Brion, Lawrence, and Varchenko on rational generating functions for cones* Math. Intelligencer **31** 1(2009), 9–17; [arXiv:math/0506466].
- [2] A. Berele, A. Regev, *Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras*, Adv. Math., **64**, (1987), 118–175.
- [3] I.N. Bernstein, I. M. Gelfand, S. I. Gelfand, *Differential operators on the base affine space and a study of  $\mathfrak{g}$ -modules*, I.M. Gelfand (ed.), Lie groups and their representations, Proc. Summer School on Group Representations, Janos Bolyai Math. Soc. & Wiley (1975), 39–64.
- [4] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. Ecole Norm. Sup. **21** 4 (1988), 653–663.
- [5] Shun-Jen Cheng, Jae-Hoon Kwon, Ngau Lam, *A BGG-Type Resolution for Tensor Modules over General Linear Superalgebra*, Lett. Math. Phys., **84** 1, (2008), 75–87; [arXiv:0801.0914].
- [6] V.A. Fateev, S.L. Lukyanov, *The models of two-dimensional conformal quantum field theory with  $Z_n$  symmetry*, Int. J. Mod. Phys. **A** 3 (2), (1988), 507–520.
- [7] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Quantum continuous  $\mathfrak{gl}_\infty$ : Tensor product of Fock modules and  $W_n$  characters*, Kyoto J. Math., **51** 2, (2011) 365–392; [arXiv:1002.3113].
- [8] B. Feigin, E. Frenkel, *Quantum  $W$ -algebras and Elliptic Algebras*, Comm. Math. Phys., **178** 3, (1996), 653–677; [arXiv:q-alg/9508009].
- [9] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Quantum toroidal  $\mathfrak{gl}_1$  algebra: plane partitions*, Kyoto J. Math., **52** 3, (2012), 621–659; [arXiv:1110.5310].
- [10] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Quantum toroidal  $\mathfrak{gl}_1$  and Bethe ansatz*, J. Phys. **A** 48 24, (2015), 244001; [arXiv:1502.07194].
- [11] B. Feigin, M. Jimbo, T. Miwa, A. Odesskii, Ya. Pugai, *Algebra of screening operators for the deformed  $W_n$  algebra*, Comm. Math. Phys., **191** (3), (1998), 501–541; [arXiv:q-alg/9702029].
- [12] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi, S. Yanagida, *Kernel function and quantum algebras*, RIMS kōkyūroku **1689**, (2010), 133–152; [arXiv:1002.2485].
- [13] B. Feigin, I. Makhlin, *A Combinatorial Formula for Affine Hall-Littlewood Functions via a Weighted Brion Theorem*, Selecta Mathematica (2016) [arXiv:1505.04269].
- [14] B. Feigin, G. Mutafyan, *The Quantum Toroidal Algebra  $\widehat{\widehat{\mathfrak{gl}_1}}$ : Calculation of Characters of Some Representations as Generating Functions of Plane Partitions*, Funct. Anal. Appl., **47** 1, (2013) 50–61.
- [15] B. Feigin, G. Mutafyan, *Characters of Representations of the Quantum Toroidal Algebra  $\widehat{\widehat{\mathfrak{gl}_1}}$ : Plane Partitions with “Stands”*, Funct. Anal. Appl., **48** 1, (2014) 36–48.
- [16] B.L. Feigin, A.M. Semikhatov,  *$W_n^{(2)}$  algebras*, Nucl. Phys. B **698** (3), (2004), 409–449; [arXiv:math/0401164].

- [17] I. Gessel, G. Viennot, *Binomial Determinants, Paths, and Hook Length Formulae*, Advances in Mathematics **58**, (1985), 300-321.
- [18] V.G. Kac, S. Roan, M. Wakimoto, *Quantum Reduction for Affine Superalgebras*, Comm. Math. Phys. **241** **2**, (2003), 307-342; [arXiv:math-ph/0302015].
- [19] J. Lawrence, *Rational-function-valued valuations on polyhedra*, Discrete and computational geometry (New Brunswick, NJ, 1989/1990), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 6, Amer. Math. Soc., Providence, RI, 1991, pp. 199-208.
- [20] B. Lindström, *On the vector representation of induced matroids*, Bull. London Math. Soc. **5**, (1973), 85-90.
- [21] A. G. Khovanskii, A. V. Pukhlikov *Finitely additive measures of virtual polyhedra*, St. Petersburg Math. J. **4** **2**, (1993), 337-356.
- [22] A. G. Khovanskii, A. V. Pukhlikov *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes* St. Petersburg Math. J. **4** **4**, (1993), 789-812.
- [23] I. Makhlin, *Weyl's Formula as the Brion Theorem for Gelfand-Tsetlin Polytopes*, [arXiv:1409.7996].
- [24] E.M. Moens, J. van der Jeugt, *A determinantal formula for supersymmetric Schur polynomials*, J. Algebraic Combin. **17** **3**, (2003), 283-307.
- [25] A. Okounkov, N. Reshetikhin, C. Vafa, *Quantum Calabi-Yau and Classical Crystals* The unity of mathematics, 597618, Progr. Math., 244, Birkhuser Boston; [arXiv:hep-th/0309208].
- [26] A.N. Sergeev, *Representations of the Lie superalgebras  $\mathfrak{gl}(n, m)$  and  $Q(n)$  on the space of tensors*, Funct. Anal. Appl., **18** **1**, (1984) 70-72.
- [27] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1 Cambridge University Press (1997).
- [28] A. Tsymbaliuk *The affine Yangian of  $gl_1$  revisited*, [arXiv:1404.5240].
- [29] A.V. Zelevinsky, *Resolutions, dual pairs, and character formulas*, Funct. Anal. Appl., **21** **2**, (1987), 152-154.

LANDAU INSTITUTE FOR THEORETICAL PHYSICS, CHERNOGOLOVKA, RUSSIA  
 INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW, RUSSIA  
 LABORATORY OF MATHEMATICAL PHYSICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF  
 ECONOMICS, MOSCOW, RUSSIA  
 INDEPENDENT UNIVERSITY OF MOSCOW, MOSCOW, RUSSIA  
*E-mail address:* mbersht@gmail.com

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW, RUSSIA  
*E-mail address:* borfeigin@gmail.com

MOSCOW CENTER FOR CONTINUOUS MATHEMATICAL EDUCATION, MOSCOW, RUSSIA  
*E-mail address:* merzon@mccme.ru